Last time

## Maxwell Equations (cont'd)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{D} = \rho & \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \end{array}}$$

in medium (1865)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} \end{array}}$$

in vacuum

$$c^2 = \frac{1}{\mu_0\varepsilon_0}$$

## Vector and Scalar Potentials (cont'd)

$$\boxed{\begin{array}{l} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial \vec{A}}{\partial t} \end{array}}$$

$\Rightarrow$ Maxwell equations become

$$\boxed{\begin{array}{l} \nabla^2\Phi + \dfrac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = -\rho/\varepsilon_0 \\ \left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\vec{A} - \vec{\nabla}\cdot\left[\vec{\nabla}\cdot\vec{A} + \dfrac{1}{c^2}\dfrac{\partial \Phi}{\partial t}\right] = -\mu_0\vec{J} \end{array}}$$

Gauge-invariance:

$$\boxed{\begin{array}{l} \vec{A} \to \vec{A} + \vec{\nabla}\Lambda \\ \Phi \to \Phi - \dfrac{\partial \Lambda}{\partial t} \end{array}}$$

leave $\vec{E}, \vec{B}$ the same.

(I) Lorenz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0$

$$\Rightarrow \boxed{\begin{aligned} \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi &= -\rho/\varepsilon_0 \\ \underbrace{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]}_{\text{wave equation operator}}\vec{A} &= -\mu_0 \vec{J} \end{aligned}}$$

Maxwell equ's in Lorenz gauge

source

In vacuum $\vec{D} = \varepsilon_0 \vec{E}$ , $\vec{B} = \mu_0 \vec{H}$ and

Maxwell equations read

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

(Vacuum!)

Here $c^2 = \dfrac{1}{\mu_0 \varepsilon_0}$ , $c \sim$ speed of light

$\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$ satisfy the last two equations. Plug into the first two:

$$\begin{cases} \nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\cdot\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \vec{J} \end{cases}$$

A somewhat more compact way of writing Maxwell equations in vacuum.

Now, $\vec{A}$ and $\Phi$ are <u>not</u> uniquely defined: one can always redefine

$$\boxed{\begin{array}{l} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{array}}$$

gauge transformations

without changing

$\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

Electrodynamics is the first gauge theory known to people.

$\Rightarrow$ By choosing various scalar functions $\Lambda(\vec{x},t)$ can satisfy different gauge conditions:

(I) Lorenz gauge: $\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$

Does it exist? Start from some random $\Phi, \vec{A}$.

Is there $\Lambda(\vec{x},t)$ such that $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0$ ?

$$\vec{\nabla}\cdot\left[\vec{A} + \vec{\nabla}\Lambda\right] + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\Phi - \frac{\partial \Lambda}{\partial t}\right] = 0$$

$$\Rightarrow \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

$\Rightarrow$ can always solve this to find $\Lambda(\vec{x},t)$

$\Lambda$ is not unique: can alwayse shift $\Lambda \to \Lambda + \tilde{\Lambda}$

where $\nabla^2\tilde{\Lambda} - \frac{1}{c^2}\frac{\partial^2\tilde{\Lambda}}{\partial t^2} = 0$ $\Rightarrow$ wave in empty space $\Rightarrow \tilde{\Lambda} \neq 0$ (does not have to be $\emptyset$). $\Rightarrow$ Lorenz gauge condition is not fully restrictive, there is a residual gauge freedom left, but that's Ok, it happens with most gauges.

Using Lorentz gauge condition we rewrite Maxwell equations as

$$\boxed{\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}}$$

Maxwell equs in Lorentz gauge

(II) Another interesting gauge is Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A} = 0$$

Coulomb gauge condition

Maxwell equations become:

$$\boxed{\begin{aligned} \nabla^2 \Phi &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} \end{aligned}}$$

Maxwell equs in Coulomb gauge

$$\Rightarrow \quad \Phi_{(\vec{x},t)} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

just like in statics

Note: time is the same on both sides, interaction is "instantaneous" $\Rightarrow$ hence the name Coulomb gauge.

$\Rightarrow$ causality is not violated, $\vec{E}$ & $\vec{B}$ propagate with the speed of light ($\Phi$ is not an observable!)

III) Other notable gauges: $x_\mu A^\mu = 0$ (Schwinger gauge),

$$A_0 + A_z = 0 \Rightarrow \Phi + c A_z = 0 \quad \text{(light cone gauge)}$$

$$A_0 = \Phi = 0 \quad \text{(axial gauge)}$$

## Green Function for Wave Equation.

In solving Maxwell equations in, say, Lorentz gauge one often encounters equations of the type:

$$\boxed{\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x},t)}$$ Inhomogeneous wave eqn.

with $f(\vec{x},t)$ some known function (source).
The strategy for solving those is the same as in electrostatics: find the Green function.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

Then $\Psi(\vec{x},t) = \int d^3x'\,dt'\, G(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')$

is a solution of the inhomogeneous wave equation. (in unlimited space-time).

In empty space $G(\vec{x},t;\vec{x}',t') \equiv G(\vec{x}-\vec{x}', t-t') \Rightarrow$ need to solve



$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r},t) = -4\pi\, \delta^3(\vec{r})\, \delta(t)$$

write $G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}\, \tilde{G}(\vec{k},\omega)$

as $\delta^3(\vec{r})\,\delta(t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$

$$\Rightarrow \left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right)\tilde{G} = -4\pi$$

$$\Rightarrow \tilde{G} = \frac{-4\pi}{\frac{\omega^2}{c^2} - \vec{k}^2}$$ photon propagator

$$\Rightarrow \boxed{G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}\, \frac{-4\pi}{\frac{\omega^2}{c^2} - \vec{k}^2}}$$

Does this expression make sense? Essential sing. at $\frac{\omega}{c} = |\vec{k}|$. There are several ways to regulate it:

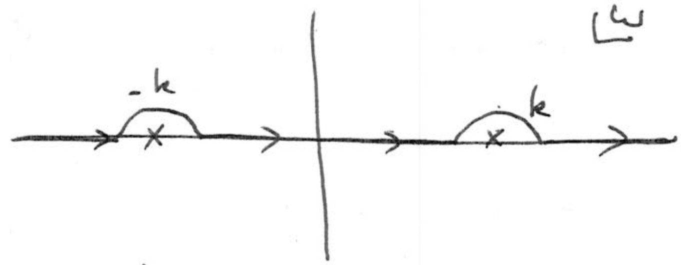
## A. Retarded (causal) Green function

demand that $G(\vec{r},t) = 0$ for $t < 0$

$$G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}}\, \frac{-4\pi}{\left(\frac{\omega}{c} - k\right)\left(\frac{\omega}{c} + k\right)}$$ with $k = |\vec{k}|$

Need $G = 0$ for $t<0$: if $t<0$ have to close the $\omega$-contour into the upper half-plane:

$\Rightarrow$ need to have poles in the lower half-plane:




$$\Rightarrow \quad G_{ret}(\vec r,t) = -4\pi \int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\, \frac{1}{\left(\frac{\omega}{c}-k+i\varepsilon\right)\left(\frac{\omega}{c}+k+i\varepsilon\right)}$$

$$\Rightarrow \quad \boxed{G_{ret}(\vec r,t) = -4\pi\int\frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\,\frac{1}{\frac{\omega^2}{c^2}-k^2+i\varepsilon\omega}}$$

advanced ~ change signs of $i\varepsilon$'s.

Do the Fourier transform

$$G_{ret}(\vec r,t) = -4\pi\,\theta(t)\,(-2\pi i)\,\frac{c^{\cancel{2}}}{\cancel{2\pi}}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot\vec r}\left[\frac{e^{-ikct}}{2k\cancel{c}} - \frac{e^{ikct}}{2k\cancel{c}}\right] = 2\pi i\,\theta(t)\, c\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot\vec r}\,\frac{1}{k}\left[e^{-ikct}-e^{ikct}\right]$$

$$= \left|\vec r \parallel \hat z\right| = \frac{2\pi i c\,\theta(t)}{(2\pi)^3}\int_0^\infty dk\cdot k^{\cancel{2}}\int_{-1}^{1} d\cos\theta\,\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\, e^{ikr\cos\theta}\,\frac{1}{\cancel{k}}\left[e^{-ikct}-e^{ikct}\right] = \frac{\cancel{i}c}{2\pi}\,\theta(t)\int_0^\infty dk\cdot\cancel{k}\cdot\frac{1}{\cancel{ik}r}\left[e^{ikr}-e^{-ikr}\right]$$

$$\cdot\left[e^{-ikct}-e^{ikct}\right]=\frac{c}{2\pi r}\theta(t)\int_0^\infty dk\left[e^{ik(r-ct)}+\right.$$

$$\left.+e^{-ik(r-ct)}-e^{-ik(r+ct)}-e^{ik(r+ct)}\right]\cdot e^{-\delta k}$$

($\delta$ is some regulator at $k\to+\infty$)

$$=\frac{c}{2\pi r}\theta(t)\cdot\left\{\frac{-1}{i(r-ct)-\delta}+\frac{-1}{-i(r-ct)-\delta}-\frac{-1}{-i(r+ct)-\delta}-\right.$$

$$\left.-\frac{-1}{+i(r+ct)-\delta}\right\}=\frac{ci}{2\pi r}\theta(t)\left\{\frac{1}{r-ct+i\delta}-\frac{1}{r-ct-i\delta}+\right.$$

$$\left.+\frac{1}{r+ct-i\delta}-\frac{1}{r+ct+i\delta}\right\}=\left|\begin{array}{l}\text{using}\\ \underbrace{\frac{1}{x-i\delta}-\frac{1}{x+i\delta}}_{\text{regulator}}=2\pi i\underbrace{\delta(x)}_{\text{Dirac delta-ftn.}}\end{array}\right.$$

do not confuse!

$$=\frac{ci}{2\pi r}\theta(t)\left\{-2\pi i\,\delta(r-ct)+\underbrace{2\pi i\,\delta(r+ct)}_{=0\text{ as }r>0,\,t>0}\right\}$$

$$=\frac{c}{r}\theta(t)\,\delta(r-ct)=\frac{1}{r}\delta\left(t-\frac{r}{c}\right)\quad\text{as desired!}$$

$$\Rightarrow\boxed{G_{ret}(\vec r,t)=\frac{1}{r}\delta\left(t-\frac{r}{c}\right)}$$

B. Advanced Green function (can be evaluated in a similar way)

$$G_{adv}(\vec r,t)=-4\pi\int\frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}e^{-i\omega t+i\vec k\cdot\vec r}\frac{1}{\frac{\omega^2}{c^2}-k^2-i\omega\varepsilon}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t - \frac{r}{c}\right)} \qquad \text{or}$$

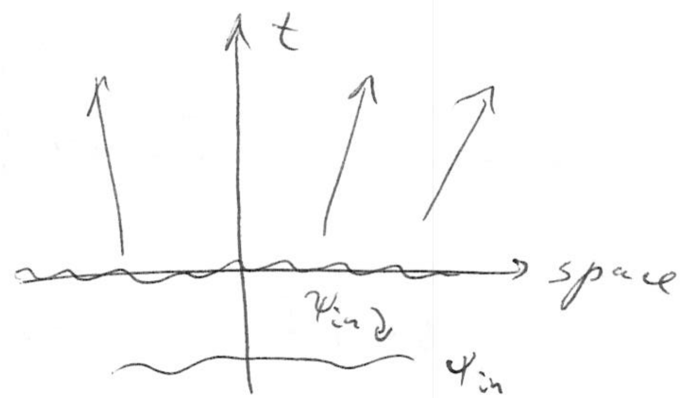
$$\boxed{G_{ret}(\vec{x}, t; \vec{x}\,', t') = \frac{1}{|\vec{x} - \vec{x}\,'|}\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}\,'|}{c}\right)}$$

Retarded Green function

Given the source (localized in space-time) $f(\vec{x}, t)$ and initial condition $\Psi_{in}(\vec{x}, t)$ ← satisfying homog. eq., $\Box \Psi_{in} = 0$. at $t = -\infty$ we can write the solution

$$\Psi(\vec{x}, t) = \Psi_{in}(\vec{x}, t) + \int d^3x'\, dt'\, G_{ret}(\vec{x}, t; \vec{x}\,', t')\, f(\vec{x}\,', t')$$

Retarded Green ftn is causal ~ gives the solution in the future due to sources in the past.




Advanced Green function

$$G_{adv}(\vec{r}, t) = G^*_{ret}(-t, -\vec{r}) \Rightarrow \text{as } |\vec{r}| = |-\vec{r}|$$

$$\Rightarrow G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(-t - \frac{r}{c}\right) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)$$

$$\Rightarrow \boxed{G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)}$$