Last time

Green function for Wave Equation

(cont'd)

$$\boxed{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi_{(\vec{x},t)} = -4\pi f(\vec{x},t)}$$ wave equation

$$\psi(\vec{x},t) = \int d^3x'\, dt'\, G(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')$$ ← solution

↑ Green function, $G = G(\vec{x}-\vec{x}', t-t')$

$$\boxed{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\vec{r},t) = -4\pi\, \delta^3(\vec{r})\, \delta(t)}$$

↑ equation for the Green function

A. Retarded Green function

$$\boxed{G_{ret}(\vec{r},t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}}\, \frac{1}{\frac{\omega^2}{c^2} - \vec{k}^2 + i\varepsilon\omega}}$$

↑ Fourier decomposition

$\Rightarrow$ integrated to obtain $\boxed{G_{ret}(\vec{r},t) = \frac{1}{r}\,\delta\left(t - \frac{r}{c}\right)}$

$$\cdot \left[e^{-ikct} - e^{ikct} \right] = \frac{c}{2\pi r} \theta(t) \int_0^\infty dk \left[e^{ik(r-ct)} + \right.$$

$$\left. + e^{-ik(r-ct)} - e^{-ik(r+ct)} - e^{ik(r+ct)} \right] \cdot e^{-\delta k}$$

(δ is some regulator at $k \to +\infty$)

$$= \frac{c}{2\pi r} \theta(t) \cdot \left\{ \frac{-1}{i(r-ct)-\delta} + \frac{-1}{-i(r-ct)-\delta} - \frac{-1}{-i(r+ct)-\delta} - \right.$$

$$\left. - \frac{-1}{+i(r+ct)-\delta} \right\} = \frac{ci}{2\pi r} \theta(t) \left\{ \frac{1}{r-ct+i\delta} - \frac{1}{r-ct-i\delta} + \right.$$

$$\left. + \frac{1}{r+ct-i\delta} - \frac{1}{r+ct+i\delta} \right\} = \left| \begin{array}{l} \text{using} \\ \frac{1}{x-i\delta} - \frac{1}{x+i\delta} = 2\pi i \, \delta(x) \end{array} \right.$$

regulator — Dirac delta-ftn. — do not confuse!

$$= \frac{ci}{2\pi r} \theta(t) \left\{ -2\pi i \, \delta(r-ct) + 2\pi i \, \underbrace{\delta(r+ct)}_{=0 \text{ as } r>0,\, t>0} \right\}$$

$$= \frac{c}{r} \theta(t) \, \delta(r-ct) = \frac{1}{r} \delta\left(t - \frac{r}{c}\right) \quad \text{as desired!}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r} \delta\left(t - \frac{r}{c}\right)}$$

B. Advanced Green function (can be evaluated in a similar way)

$$G_{adv}(\vec{r}, t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} e^{-i\omega t + i\vec{k}\cdot\vec{r}} \frac{1}{\frac{\omega^2}{c^2} - k^2 - i\omega\varepsilon}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t - \frac{r}{c}\right)} \quad \text{or}$$

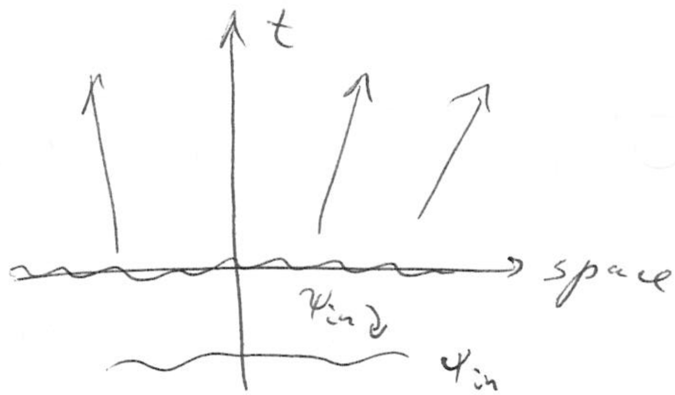
$$\boxed{G_{ret}(\vec{x}, t; \vec{x}', t') = \frac{1}{|\vec{x} - \vec{x}'|}\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)}$$

Retarded Green function

Given the source (localized in space-time) $f(\vec{x}, t)$ and initial condition $\psi_{in}(\vec{x}, t)$ (satisfying homog. eq., $\Box\psi_{in} = 0$. at $t = -\infty$) we can write the solution

$$\psi(\vec{x}, t) = \psi_{in}(\vec{x}, t) + \int d^3x'\,dt'\,G_{ret}(\vec{x}, t; \vec{x}', t')\,f(\vec{x}', t').$$

Retarded Green ftn is causal ~ gives the solution in the future due to sources in the past.



<u>Advanced Green function</u>:

$$G_{adv}(\vec{r}, t) = G^*_{ret}(-t, -\vec{r}) \Rightarrow \text{as } |\vec{r}| = |-\vec{r}|$$

$$\Rightarrow G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(-t - \frac{r}{c}\right) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)$$

$$\Rightarrow \boxed{G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)}$$

Therefore

$$\boxed{G_{adv}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\!\left(t-t'+\frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

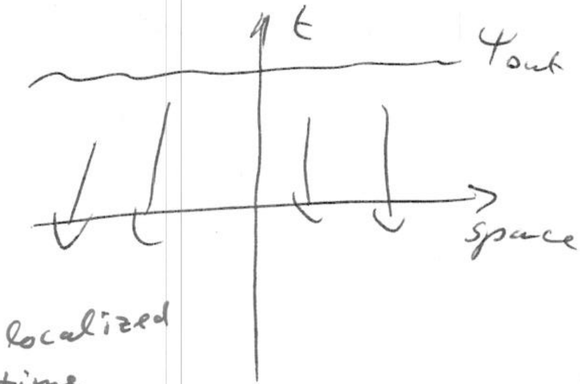
Work opposite to G_{ret} ~ acausal:

also satisfies $\Box\,\psi_{out}=0$.

$$\psi(\vec{x},t) = \psi_{out}(\vec{x},t) +$$

$$+\int d^3x'\,dt'\; G_{adv}(\vec{x},t;\vec{x}',t')\,f(\vec{x}',t').$$

source is localized in space-time



Solution of Maxwell equations in Lorenz gauge:

$$\begin{cases} \nabla^2\Phi - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\Phi = -\dfrac{\rho}{\varepsilon_0} \\[2ex] \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{J} \end{cases}$$

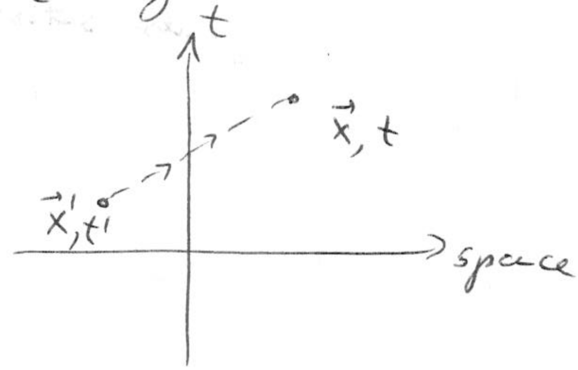
$\Rightarrow$ assume $\Phi_{in}=0$ $\vec{A}_{in}=0$ and use retarded Green ftn.

$$\Rightarrow\ \boxed{\begin{aligned} \Phi(\vec{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\!\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\!\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}}$$

where we integrated over t' the δ-ftn:

$$\delta\!\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) \text{ to get } t' = t-\frac{|\vec{x}-\vec{x}'|}{c}.$$

Physical meaning of $t' = t - \frac{|\vec{x} - \vec{x}\,'|}{c}$: for a source at time t' to affect the field at time t they need to be $c(t-t')$ away from each other ~ just far enough for light to travel!

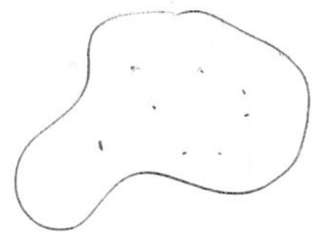


Poynting's Theorem and Conservation of Energy and Momentum.

Energy:

Consider several point charges $q_1, \ldots, q_N$ located at $\vec{x}_1, \ldots, \vec{x}_N$ & moving with velocities $\vec{v}_1, \ldots, \vec{v}_N$ in external electromagnetic field:

The work on these charges due to EM field per unit time is



$$\sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n = \sum_{n=1}^{N} q_n \vec{E}(\vec{x}_n, t) \cdot \vec{v}_n =$$

$$= \int d^3x \left(\sum_{n=1}^{N} q_n \vec{v}_n \delta(\vec{x} - \vec{x}_n) \right) \cdot \vec{E}(\vec{x}, t) =$$

$$= \int d^3x \; \vec{J} \cdot \vec{E}$$ ~ work due to $\vec{E}$-field on the charges $\vec{J}$

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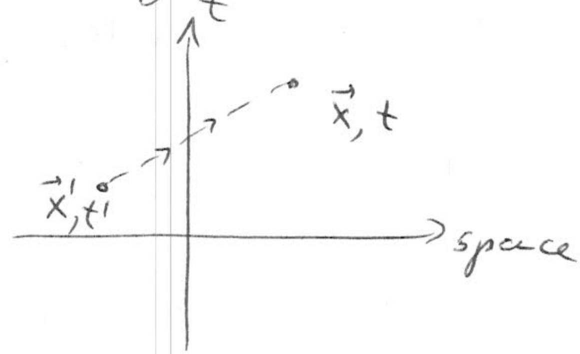

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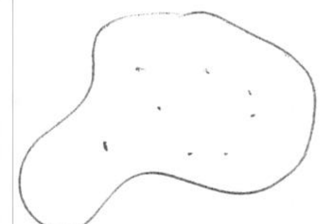

The work on these charges due to EM field per unit time is

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$$= \int d^3x \left(\sum_{n=1}^{N} q_n \vec{v}_n \, \delta(\vec{x} - \vec{x}_n) \right) \cdot \vec{E}(\vec{x}, t) =$$

$$= \int d^3x \; \vec{J} \cdot \vec{E}$$ ~ work due to net $\vec{E}$ - field on the current $\vec{J}$

Using Maxwell equation resulting from modifying Ampere's law, $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, write

$$\int d^3x \; \vec{J} \cdot \vec{E} = \int d^3x \; \vec{E} \cdot \left(\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right)$$

as $\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$

$$\Rightarrow \int d^3x \; \vec{J} \cdot \vec{E} = - \int d^3x \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \underbrace{(\vec{\nabla} \times \vec{E})}_{= -\partial \vec{B} / \partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] = - \int d^3x \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

Definition Define $\boxed{u \equiv \frac{1}{2} \left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right)}$

If we have a linear medium (e.g. $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$) or vacuum ($\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$), then u has the meaning of <u>energy density</u> of EM fields.

$\Rightarrow$ we get $\boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{J} \cdot \vec{E}}$

"−" as $\vec{J} \cdot \vec{E}$ is the work done on the charges

Looks like continuity relation ...

$\Rightarrow$ Definition Define Poynting vector (energy flow)

$$\boxed{\vec{S} \equiv \vec{E} \times \vec{H}}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E}}$$ Statement of energy conservation.

$u \sim$ energy density of EM fields $\Rightarrow u \to u_{field}$

$\vec{J} \cdot \vec{E} \sim$ rate of EM energy change due to work done on charges: $\vec{J} \cdot \vec{E} = \frac{\partial u_{mech}}{\partial t}$ ← mechanical energy density

(momentum density)

$\vec{S} \sim$ flow of energy in/out of the system:

$$\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} = - \vec{\nabla} \cdot \vec{S}$$

$\Rightarrow$ integrate over V:

$$\frac{\partial E_{field}}{\partial t} + \frac{\partial E_{mech}}{\partial t} = - \oint_S da \cdot S_n$$ ~ flow of energy through the border.

(we assume that no particles move in/out of V!)

Momentum: force on a charge is

$\vec{F} = q\left(\vec{E} + \vec{V} \times \vec{B}\right) \Rightarrow$ "force density"

is $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \Rightarrow$ the change per unit time of the total momentum of all the particles $\vec{P}_{mech}$ is

$$\frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\rho \vec{E} + \vec{J} \times \vec{B} \right].$$

Now, let's work in vacuum: $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\varepsilon_0 \vec{E}(\vec{\nabla}\cdot\vec{E}) + \varepsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} - \right.$$

$$\left. - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] = \int d^3x \; \varepsilon_0 \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + E \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{\substack{= -\vec{\nabla}\times\vec{E} \\ \text{(Faraday)}}} - \right.$$

$$\left. - \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] =$$

$$= \varepsilon_0 \int d^3x \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2 \vec{B} \cdot \underbrace{(\vec{\nabla}\cdot\vec{B})}_{\overset{\text{}}{= 0 \text{ can add}}} - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] - \varepsilon_0 \int d^3x \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

Defining $\vec{P}_{field} = \varepsilon_0 \int d^3x \; \vec{E} \times \vec{B} = \frac{1}{c^2} \int d^3x \; \vec{E} \times \vec{H} = \frac{1}{c^2} \int \underbrace{\vec{S}}_{\text{Poynting vector}} d^3x$

we write

$$\frac{d}{dt}\vec{P}_{field} + \frac{d}{dt}\vec{P}_{mech} = \varepsilon_0 \int d^3x \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2 \vec{B}(\vec{\nabla}\cdot\vec{B}) - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]$$