Last time | Solved Maxwell equations in Lorenz gauge:

$$\boxed{\begin{aligned}\Phi(\vec{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)\\ \vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)\end{aligned}}.$$

Poynting's Theorem & Conservation of Energy and Momentum (cont'd)

Def. $u \equiv \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)$ field energy density

Def. $\vec{S} \equiv \vec{E}\times\vec{H}$ Poynting vector (flow of energy)

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$ energy conservation

$$\frac{\partial u_{mech}}{\partial t} = \vec{J}\cdot\vec{E} \Rightarrow \boxed{\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}.$$

$\Rightarrow$ Definition Define Poynting vector (energy flow)

$$\boxed{\vec{S} \equiv \vec{E} \times \vec{H}}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E}}$$

Statement of energy conservation.

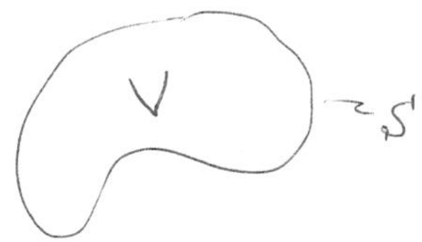
$u \sim$ energy density of EM fields $\Rightarrow u \to u_{field}$

$\vec{J} \cdot \vec{E} \sim$ rate of EM energy change due to work done on charges: $\vec{J} \cdot \vec{E} = \frac{\partial u_{mech}}{\partial t}$ $\leftarrow$ mechanical energy density

(momentum density)

$\vec{S} \sim$ flow of energy in/out of the system:

$$\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} = - \vec{\nabla} \cdot \vec{S}$$



$\Rightarrow$ integrate over V:

$$\frac{\partial E_{field}}{\partial t} + \frac{\partial E_{mech}}{\partial t} = - \oint_S da \cdot S_n$$

$\sim$ flow of energy through the border.

(we assume that no particles move in/out of V!)

Momentum: force on a charge is

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \Rightarrow \text{"force density"}$$

is $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \Rightarrow$ the change per unit time of the total momentum of all the particles $\vec{P}_{mech}$ is

$$\frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\rho \vec{E} + \vec{J} \times \vec{B} \right].$$

Now, let's work in vacuum: $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\varepsilon_0 \vec{E} (\vec{\nabla} \cdot \vec{E}) + \varepsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} - \right.$$

$$\left. - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] = \int d^3x \; \varepsilon_0 \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) + E \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{= -\vec{\nabla} \times \vec{E}} - \right.$$

(Faraday)

$$\left. - \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] =$$

$$= \varepsilon_0 \int d^3x \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) + c^2 \vec{B} \cdot \overbrace{(\vec{\nabla} \cdot \vec{B})}^{= 0 \text{ can add}} - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] - \varepsilon_0 \int d^3x \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

Defining $\vec{P}_{field} = \varepsilon_0 \int d^3x \; \vec{E} \times \vec{B} = \frac{1}{c^2} \int d^3x \, \vec{E} \times \vec{H} = \frac{1}{c^2} \int \vec{S} \, d^3x$

Poynting vector

we write

$$\frac{d}{dt} \vec{P}_{field} + \frac{d}{dt} \vec{P}_{mech} = \varepsilon_0 \int d^3x \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) + c^2 \vec{B} (\vec{\nabla} \cdot \vec{B}) - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]$$

$$\vec{E} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E}) - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

$$\Rightarrow \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right]_i = E_i \nabla_j E_j -$$

$$- \frac{1}{2} \nabla_i (E_j E_j) + E_j \nabla_j E_i = \nabla_j \left[E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2 \right]$$

Hence $\qquad i, j = 1, 2, 3$

$$\left(\frac{d \vec{P}_{field}}{dt} + \frac{d \vec{P}_{mech}}{dt} \right)_i = \varepsilon_0 \int d^3x \; \nabla_j \left[E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2 + \right.$$

$$\left. + c^2 \left(B_i B_j - \frac{1}{2} \delta_{ij} \vec{B}^2 \right) \right] \sim$$ surface term, responsible for ~~momentum~~ momentum flow through the boundary

Definition Define Maxwell stress tensor (related to energy-momentum tensor): $\qquad i, j = 1, 2, 3$

$$\boxed{T_{ij} = \varepsilon_0 \left[E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2 + c^2 B_i B_j - \frac{c^2}{2} \delta_{ij} \vec{B}^2 \right]}$$

Then

$$\boxed{\frac{d}{dt} \left(\vec{P}_{field} + \vec{P}_{mech} \right)_i = \int_V d^3x \; \nabla_j T_{ij} = \oint_S da \cdot n_j T_{ij}}$$

$$\operatorname{Tr} T_{ij} = T_{ii} = \varepsilon_0 \left[-\frac{1}{2} \vec{E}^2 - \frac{c^2}{2} \vec{B}^2 \right] = -\frac{1}{2} \vec{D} \cdot \vec{E} - \frac{1}{2} \vec{B} \cdot \vec{H} =$$

$= -u$. $\sim$ energy density.

Energy - Momentum tensor:

we studied: u ~ energy density

$\vec{S}$ ~ Poynting vector (energy flow)

T_{ij} ~ Maxwell stress tensor (momentum flow)

These seemingly unrelated quantities form one tensor under Lorentz transformations, called energy-momentum tensor:

$$T^{\mu\nu} = \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{11} & -T_{12} & -T_{13} \\ S_y/c & -T_{21} & -T_{22} & -T_{23} \\ S_z/c & -T_{31} & -T_{32} & -T_{33} \end{pmatrix}$$

where $\mu, \nu = 0, 1, 2, 3$

or time component: $x^0 = tc$, $x^1 = x$, $x^2 = y$, $x^3 = z$

all the above conservation laws read ($\rho = 0$, $\vec{J} = 0$ no pt. charges, can be included in $T^{\mu\nu}$)

$$\boxed{\frac{\partial}{\partial x^\mu} T^{\mu\nu} = 0} \quad \text{(summation over } \mu\text{)}$$

$T_\mu{}^\mu = u + T_i{}^i = u - u = 0$ ~ traceless

T_{11}, T_{22}, T_{33} ~ radiation pressure components

(T_{ij} is the flux of the ith component of momentum density in the $\hat{j}$ direction)

Classification of Physical Quantities by Space-Time Symmetries.

A. Spatial rotations

$i, j = 1, 2, 3$

$$x_i \to x'_i = R_{ij} x_j \quad , \quad R_{ij} \text{ - rotation matrix}$$

$$\vec{x}'^2 = \vec{x}^2 \text{ under rotation} \Rightarrow (R_{ij} x_j)(R_{ik} x_k) = x_i x_i$$

$$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \quad \Rightarrow \quad (R^{-1})_{ij} = R_{ji}$$

$\Rightarrow (\det R)^2 = 1$

$\det R = +1$ ~ rotation w/o reflection ($\det R = -1$: rot + reflection)

vectors: $\quad A_i \to A'_i = R_{ij} A_j$

tensors: $\quad T \to T'_{i_1 i_2 \dots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} \cdot T_{j_1 j_2 \dots j_n}$

(definition) $\quad n$ = rank of the tensor

B. Spatial Reflection (parity)

$\vec{x} \to -\vec{x}$ ~ all vectors (or polar vectors)

transform like this

$$\vec{z} = \vec{x} \times \vec{y} \Rightarrow \left.\begin{array}{l} \vec{x} \to -\vec{x} \\ \vec{y} \to -\vec{y} \end{array}\right\} \Rightarrow \vec{z} \to \vec{z}$$

axial vector (pseudovector)

Inversion is also called parity $\mathbb{P}$.

$\mathbb{P}$: vector $\to$ $-$vector, axial vector $\to$ axial vector

$p=-1$ $\qquad\qquad$ $p=+1$

Tensor of rank N: $\mathbb{P}\, T_{i_1 \ldots i_N} = (-1)^N T_{i_1 \ldots i_N}$

Pseudotensor of rank N: $\mathbb{P}\, T_{i_1 \ldots i_N} = (-1)^{N+1} T_{i_1 \ldots i_N}$

[E.g. $\vec{z} = \vec{x} \times \vec{y} \Rightarrow \underbrace{z_i}_{p=1} = \varepsilon_{ijk}\, \underbrace{x_j}_{p=-1}\, \underbrace{y_k}_{p=-1} \Rightarrow \varepsilon_{ijk}$ has $p=+1$

$\Rightarrow p = (-1)^{3+1} \Rightarrow \varepsilon_{ijk}$ is pseudotensor.]

pseudoscalar anyone? $\vec{a}\cdot(\vec{b}\times\vec{c})$.

C. <u>Time reversal</u>: $\boxed{t \to -t}$

$\underbrace{\mathbb{T}\vec{x} = \vec{x}}_{T\text{-even.}}$, $\vec{p} = \frac{d\vec{x}}{dt} \Rightarrow \mathbb{T}\vec{p} = -\vec{p}$ ~ T-odd.

Quantity	Tensor Rank	Parity	Time Reversal
$\vec{x}$	vector	-1	1
$\vec{v} = d\vec{x}/dt$	vector	-1	-1
$\vec{p}$	vector	-1	-1
$\vec{L} = \vec{x}\times\vec{p}$	1	1	-1
$\vec{F} = m\vec{a}$	1	-1	1
$\vec{N} = \vec{x}\times\vec{F}$	1	1	1
E ~ energy	0	1	1

Quantity	Tensor Rank	Parity	Time Reversal
ρ	0	1	1
$\vec{J}\ (=\rho\vec{v})$	1	-1	-1
$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow$			
$\left.\begin{array}{l}\vec{E}\\ \vec{P}\\ \vec{D}\end{array}\right\}$	1	-1	1
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$			
$\left.\begin{array}{l}\vec{B}\\ \vec{M}\\ \vec{H}\end{array}\right\}$	1	1	-1
$\vec{S} = \vec{E}\times\vec{H}$	1	-1	-1
T_{ij}	2	1	1