Last time | Plane Electromagnetic Waves (cont'd)

Maxwell equ's in empty space or in LIH medium gave us

$$\boxed{\begin{aligned} \left[\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right]\vec{E} &= 0 \\ \left[\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right]\vec{B} &= 0 \end{aligned}}$$

Solution:

$$\boxed{\begin{aligned} \vec{E}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{E}(\vec{k},\omega)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + \vec{E}^*(\vec{k},\omega)\, e^{i\omega t - i\vec{k}\cdot\vec{x}}\right] \\ \vec{B}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{B}(\vec{k},\omega)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + \vec{B}^*(\vec{k},\omega)\, e^{i\omega t - i\vec{k}\cdot\vec{x}}\right] \end{aligned}}$$

where $\omega = k/\sqrt{\mu\varepsilon}$.

Def.

Phase velocity $\boxed{v = \frac{\omega}{k} = \frac{c}{n}}$

$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$ = index of refraction

However, the above solution for $\vec{E}$ & $\vec{B}$ has further constraints (on $\vec{E}(\vec{k},\omega)$ and $\vec{B}(\vec{k},\omega)$) coming from the Maxwell equations.

Let's explore those using a monochromatic wave.

Plane Electromagnetic Waves

(in infinite LIH media)

no sources ⇒

$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = 0$$

$$\vec\nabla\cdot\vec D = 0 \qquad \vec\nabla\times\vec H - \frac{\partial\vec D}{\partial t} = 0$$

$$\vec D = \varepsilon\vec E,\ \vec B = \mu\vec H \Rightarrow \vec\nabla\times\vec B - \mu\varepsilon\frac{\partial\vec E}{\partial t} = 0$$

$$\Rightarrow \vec\nabla\times(\vec\nabla\times\vec B) = \vec\nabla\cdot(\underbrace{\vec\nabla\cdot\vec B}_{0}) - \nabla^2\vec B = -\nabla^2\vec B = \mu\varepsilon\frac{\partial\vec\nabla\times\vec E}{\partial t} = -\mu\varepsilon\frac{\partial^2\vec B}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec B = 0}$$

Similarly, $\boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec E = 0}$

In general, write $\vec E(\vec x,t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec E(\vec k,\omega)\,e^{i(\vec k\cdot\vec x-\omega t)}$

and $\vec B(\vec x,t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec B(\vec k,\omega)\,e^{i(\vec k\cdot\vec x-\omega t)}$

$$\Rightarrow -\vec k^2 + \mu\varepsilon\omega^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{\mu\varepsilon}}|\vec k|$$

Phase velocity of a wave $\quad V = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ Speed of wave crest

with the index of refraction $\quad n = \sqrt{\frac{\mu\,\varepsilon}{\mu_0\,\varepsilon_0}}$

$\underbrace{\vec k\cdot\vec x - \omega t}_{\text{phase}} = \text{const} \Rightarrow$ get $v = \frac{\omega}{k}\left(= \frac{d\vec x}{dt}\right)$.

General solution (for real $\vec{E}$ & $\vec{B}$):

different from $\vec{E}(\vec{k},\omega)$ on prev. page.

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \vec{E}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right] \\ \vec{B}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{B}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \underbrace{\vec{B}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}}_{\text{complex conjugate of 1st term}}\right]\end{aligned}}$$

$\omega_k = k/\sqrt{\mu\varepsilon}$

cf. in 1-dim solution of wave equation

is $u(x,t) = f(x-vt) + g(x+vt)$

(can always redefine $\vec{k} \to -\vec{k}$ in the 2nd term $\Rightarrow$ get $\omega t + \vec{k}\cdot\vec{x}$ argument)

Monochromatic plane wave: (fix one mode $\vec{k}$)

$$\begin{cases}\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t) \\ \vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{x} - \omega t)\end{cases}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$$

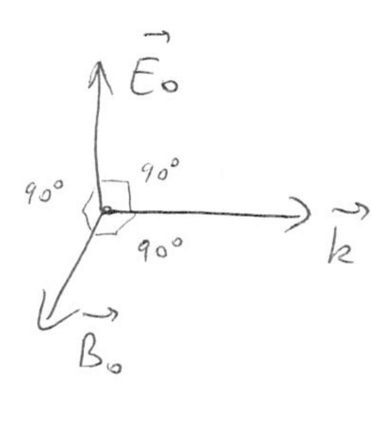
$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{E}_0 - \omega\vec{B}_0 = 0$$

$$\Rightarrow \vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$$

$$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{B}_0 + \mu\varepsilon\,\omega\vec{E}_0 = 0$$

$$\Rightarrow \vec{E}_0 = -\frac{\omega}{\vec{k}^2}\vec{k}\times\vec{B}_0$$



$\Rightarrow \vec{k}, \vec{E}_0$ & $\vec{B}_0$ are orthogonal to each other! $|\vec{B}_0| = \sqrt{\mu\varepsilon}\,|\vec{E}_0|$

Energy density

$$u = \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right) = \left[\frac{1}{2}\varepsilon E_0^2 + \right.$$

$$\left. + \frac{1}{2\mu} B_0^2\right] \cos^2(\vec{k}\cdot\vec{x} - \omega t) = \varepsilon E_0^2 \cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle u\rangle = \varepsilon E_0^2 \overbrace{\langle \cos^2(\vec{k}\cdot\vec{x} - \omega t)\rangle}^{= 1/2} =$

$= \frac{1}{2}\varepsilon E_0^2 \Rightarrow \langle u\rangle = \frac{1}{2}\varepsilon E_0^2$

Poynting vector

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}E_0 B_0 \hat{k}\cos^2(\vec{k}\cdot\vec{x} - \omega t) =$$

$$= \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \hat{k} \cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle \vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_0^2 \hat{k}$ energy flow

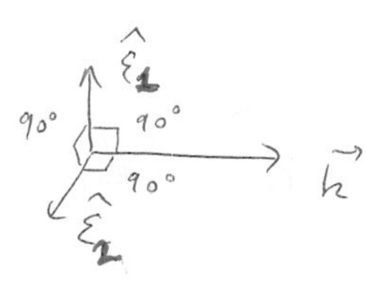
$\langle \vec{S}\rangle = \frac{1}{\sqrt{\mu\varepsilon}}\langle u\rangle \hat{k} \Rightarrow$ energy is traveling with velocity $\vec{v} = \frac{\hat{k}}{\sqrt{\mu\varepsilon}} = \frac{\omega}{k}\hat{k}$.

Polarization.

Plane wave: $\vec{E} = \mathrm{Re}\left\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}\right\}$, $\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$

in the future drop Re sign.

Choose a basis: $\hat{\varepsilon}_1$ & $\hat{\varepsilon}_2$ in the plane transverse to $\vec{k}$



$$\Rightarrow \vec{E} = (\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

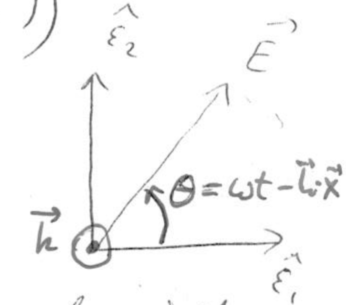
- E_1, E_2 ~ are coefficients, generally complex.

Def. if E_1 & E_2 have the same phases the wave is <u>linearly polarized</u> in the direction defined by angle $\theta = \tan^{-1}\left(\frac{E_2}{E_1}\right)$, with amplitude $E = \sqrt{|E_1|^2 + |E_2|^2}$

Suppose phases of E_1 & E_2 differ by 90°: $E_2 = \pm i E_1$

$$\Rightarrow \vec{E} = E_1 (\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \text{taking Re} =$$

$$= E_1 \left(\hat{\varepsilon}_1 \cos(\omega t - \vec{k}\cdot\vec{x}) \pm \hat{\varepsilon}_2 \sin(\omega t - \vec{k}\cdot\vec{x}) \right)$$



- the wave direction rotates with time at any fixed location $\vec{x}$.

Def. $\Rightarrow$ Such wave is called <u>circularly polarized</u>.

$\hat{\varepsilon}_1 + i\hat{\varepsilon}_2$ ~ counterclockwise ~ left circ. polar. ~ positive helicity
— clockwise ~ right — " — ~ negative helicity

Change the basis: $\hat{\varepsilon}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2)$

$$\Rightarrow \vec{E} = (E_+ \hat{\varepsilon}_+ + E_- \hat{\varepsilon}_-)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

left- & right- polarized waves

If $E_+ = \pm E_-$ $\Rightarrow$ get linearly polarized wave

Any linearly polarized wave is a superposition of 2 circularly polarized waves with opposite helicities.