Last time: monochromatic plane wave

$$\begin{cases} \vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t) \\ \vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{x} - \omega t) \end{cases} \Rightarrow \text{Maxwell eqn's give}$$

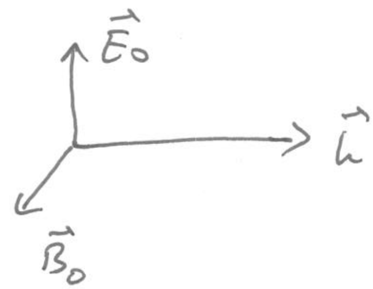
$$\vec{k}\cdot\vec{E}_0 = 0 = \vec{k}\cdot\vec{B}_0$$

$$\vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$$

$$\Rightarrow \text{as } \omega = \frac{k}{\sqrt{\mu\varepsilon}} \Rightarrow B_0 = \sqrt{\mu\varepsilon}\, E_0$$

$$\langle u \rangle = \frac{1}{2}\varepsilon E_0^2 \qquad \langle \vec{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\, E_0^2\, \hat{k}$$

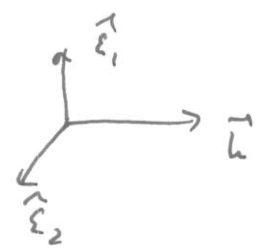
time averaged



Polarization:

$$\vec{E} = \mathrm{Re}\left\{\vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\right\}$$

$$\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$$

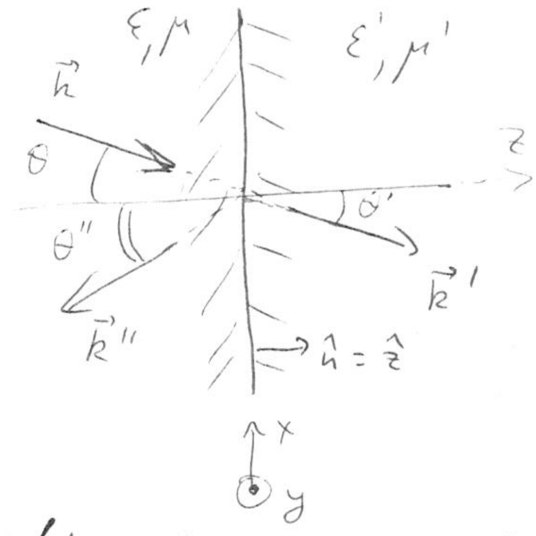


$$\vec{E} = \underbrace{(\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2)}_{\text{linear polarizations}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

$$\vec{E} = (\hat{\varepsilon}_+ E_+ + \hat{\varepsilon}_- E_-)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

$$\hat{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2) \sim \text{circular polarizations}$$

Reflection and Refraction



(does not assume that all $\vec{k}$'s are in one plane)

Incident wave:

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \\ \vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E} = \sqrt{\mu\varepsilon} \, \frac{\vec{k}\times\vec{E}}{k} \end{cases}$$

Refracted wave:

$$\begin{cases} \vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x} - \omega' t)} \\ \vec{B}' = \sqrt{\mu'\varepsilon'} \, \frac{\vec{k}'\times\vec{E}'}{k'} \end{cases}$$

Reflected wave: $\begin{cases} \vec{E}'' = \vec{E}_0'' \, e^{i(\vec{k}''\cdot\vec{x} - \omega'' t)} \\ \vec{B}'' = \sqrt{\mu\varepsilon} \, \frac{\vec{k}''\times\vec{E}''}{k''} \end{cases}$

Match boundary conditions: $\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow D_n$ is cont.

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow B_n$ is continuous

$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0 \Rightarrow E_t$ is continuous

as $\vec{B}$ has no δ-function singularity at $z=0$

$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega\vec{D} \Rightarrow H_t$ is continuous (same reason)

to have any boundary conditions at $z=0$ need

$$\boxed{\omega = \omega' = \omega''} \Rightarrow \boxed{k = k'' = \sqrt{\mu\varepsilon}\,\omega, \quad k' = \sqrt{\mu'\varepsilon'}\,\omega}$$

spatial phase factors should also be equal

at $z=0$: $\vec{k}\cdot\vec{x}\big|_{z=0} = \vec{k}'\cdot\vec{x}\big|_{z=0} = \vec{k}''\cdot\vec{x}\big|_{z=0}, \ \forall x, y$

$\Rightarrow$ choose $\vec{k} = (k_x, 0, k_z) \Rightarrow \vec{k}\cdot\vec{x}\big|_{z=0} = k_x \cdot X \Rightarrow$ no y-dep

$\Rightarrow$ there should be no y-dependence in $\vec{k}'\cdot\vec{x}$ and in $\vec{k}''\cdot\vec{x}$ as well $\Rightarrow k_y' = k_y'' = 0 \Rightarrow$ all lie in the <u>same plane</u>

$$k \cdot \sin\Theta = k' \cdot \sin\Theta' = k'' \cdot \sin\Theta''$$

$\Rightarrow$ as $k = k'' \Rightarrow$ $\Theta = \Theta''$ $\approx$ angle of reflection is equal to angle of incidence!

as $k = \sqrt{\mu\varepsilon}\,\omega$ and $k' = \sqrt{\mu'\varepsilon'}\,\omega \Rightarrow$

$\sqrt{\mu\varepsilon}\,\sin\theta = \sqrt{\mu'\varepsilon'}\,\sin\Theta'$. Remember $n = c\sqrt{\mu\varepsilon}$ (index of refraction) $\Rightarrow$ $n\sin\Theta = n'\sin\theta'$

Snell's law!

The only thing left is to find $\vec{E}_0'$ & $\vec{E}_0''$ using b.c.'s:

D_n $(= \varepsilon E_n)$ continuous $\Rightarrow \hat{n}\cdot\left[\varepsilon\left(\vec{E}_0 + \vec{E}_0''\right) - \varepsilon'\vec{E}_0'\right] = 0$

B_n continuous $\Rightarrow \hat{n}\cdot\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'' - \vec{k}'\times\vec{E}_0'\right] = 0$
(and $\omega = \omega' = \omega''$)

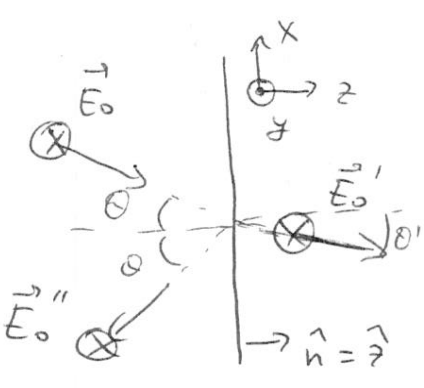
E_t continuous $\Rightarrow \hat{n}\times\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right] = 0$

H_t continuous $\Rightarrow \left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right) - \frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n} = 0$

Consider 2 cases: (linear polarization)

(I) <u>$\vec{E}_0 \perp$ to the plane of incidence</u>

$\vec{E}_0, \vec{E}_0', \vec{E}_0'' \parallel \hat{y}$



3rd & 4th equations: $(\hat{n} = \hat{z})$

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \frac{1}{\mu}\left(k E_0 \cos\theta - k'' E_0'' \cos\theta''\right) - \frac{1}{\mu'} k' E_0' \cos\theta' = 0 \end{cases}$$

as $k = k'' = \sqrt{\mu\varepsilon}\,\omega$, $k' = \sqrt{\mu'\varepsilon'}\,\omega$ $\Rightarrow$ and $\theta = \theta''$

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \sqrt{\frac{\varepsilon}{\mu}}\left(E_0 - E_0''\right)\cos\theta - \sqrt{\frac{\varepsilon'}{\mu'}} \cdot E_0' \cdot \cos\theta' = 0 \end{cases}$$

1st equ. $0 = 0$; 2nd equ.: $k E_0 \sin\theta + k'' E_0'' \sin\theta'' - k' E_0' \sin\theta' = 0 \Rightarrow \left(E_0 + E_0''\right)\sin\theta - \sqrt{\frac{\mu'\varepsilon'}{\mu\varepsilon}}\, E_0' \sin\theta' = 0$

as $\sqrt{\mu\varepsilon}\sin\theta = \sqrt{\mu'\varepsilon'}\sin\theta'$ (Snell's law) $\Rightarrow E_0 + E_0'' - E_0' = 0$

$\Rightarrow$ duplicates the 3rd one.

Using Snell's law ($n \sin\theta = n' \sin\theta'$) to get rid of n' we write (work it out yourself):

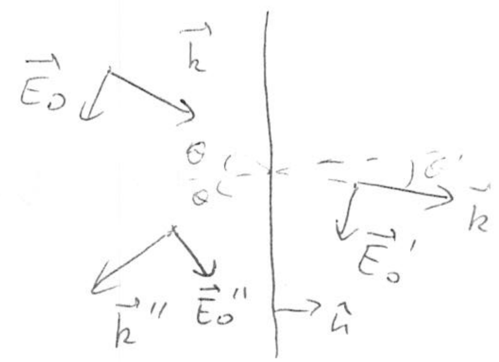
$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

$n' = \sqrt{\frac{\mu'\varepsilon'}{\mu_0\varepsilon_0}}$

II $\vec{E}_0 \parallel$ plane of incidence (xz plane)

2 independent equations (3rd & 4th):



$$\begin{cases} (E_0 + E_0'')\cos\theta - E_0'\cos\theta' = 0 \\ \sqrt{\frac{\varepsilon}{\mu}}\,(E_0 - E_0'') - \sqrt{\frac{\varepsilon'}{\mu'}}\,E_0' = 0 \end{cases}$$

(other two can be reduced to those)

Solve: (using Snell's Law)

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{-\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

Normal incidence: $\theta = 0 \Rightarrow$ both I and II give

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'}n'}, \quad \frac{E_0''}{E_0} = \frac{n - \frac{\mu}{\mu'}n'}{n + \frac{\mu}{\mu'}n'}$$

if $\mu = \mu'$ & $n' > n$ $\Rightarrow$ reflected wave change sign - phase reverse

Polarization by reflection: put $\mu = \mu'$ for simplicity

I: $$\frac{E_0''}{E_0} = \frac{n\cos\theta - \sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}}$$ different $\Rightarrow$

II: $$\frac{E_0''}{E_0} = \frac{-n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$ $\Rightarrow$ reflected light is polarized!

in case (I) $\frac{E_0''}{E_0}$ never vanishes (always < 0 if $n' > n$)

in case (II) $\frac{E_0''}{E_0} = 0$ for $\boxed{\theta_B = \tan^{-1}\left(\frac{n'}{n}\right)}$ Brewster's angle

$$\left[\begin{array}{l} n^2 n'^2 - n^4 \sin^2\theta = n'^4 \cos^2\theta \Rightarrow \\ \Rightarrow n^2 n'^2 (1+\tan^2\theta) - n^4 \tan^2\theta = n'^4 \Rightarrow n^2 n'^2 - n'^4 = n^2(n^2 - n'^2)\tan^2\theta \Rightarrow \tan^2\theta = \frac{n'^2}{n^2} \end{array}\right]$$

$\Rightarrow$ reflected light is <u>polarized</u>.

if $\theta = \theta_B \Rightarrow$ polarization is linear, $\perp$ to the plane of incidence.

(fish in the ocean reflect light ~ squids with polarized vision can see them)

<u>Total internal reflection</u>: Snell's law:

$$n \sin\theta = n' \sin\theta' \leq n' \Rightarrow \theta \leq \sin^{-1}\left(\frac{n'}{n}\right) \Rightarrow$$

$\Rightarrow$ for $\boxed{\theta > \sin^{-1}\left(\frac{n'}{n}\right)}$ get total reflection "evanescent wave"

$$k' = \sqrt{\mu'\varepsilon'}\,\omega \Rightarrow k'_x = -\sqrt{\mu'\varepsilon'}\,\omega \sin\theta' = -\sqrt{\mu_0\varepsilon_0}\,\omega\, n' \sin\theta' = -\frac{\omega}{c} n \sin\theta$$

$$\overset{\shortparallel}{n'\frac{\omega}{c}} \qquad k'_y = 0$$

$$k'_z = \sqrt{k'^2 - k'^2_x} = \sqrt{n'^2 - n^2 \sin^2\theta}\;\frac{\omega}{c}$$

$\Rightarrow$ for $\theta > \sin^{-1}\left(\frac{n'}{n}\right)$: k'_z becomes imaginary $k'_z = +i|k'_z|$

$$\Rightarrow e^{i k'_z z} \sim e^{-|k'_z| z} \sim \text{exponential falloff}$$

effectively the wave gets reflected from a different surface ~ violation of geom. optics, Goos-Hänchen effect

transmission coefficient $\vec{S} = \vec{E}\times\vec{H} = \mathrm{Re}[\vec{E}\times\vec{H}^*]$

$$T = \frac{|\vec{S}'|}{|\vec{S}|} = \frac{E_0'\cdot H_0'\cdot \frac{1}{2}}{E_0\cdot H_0\cdot \underset{\uparrow \langle\cos^2\rangle \sim \text{phase}}{\frac{1}{2}}} = \frac{\mu}{\mu'}\frac{E_0'\cdot B_0'}{E_0 B_0} = \frac{\mu}{\mu'}\frac{\sqrt{\mu'\varepsilon'}}{\sqrt{\mu\varepsilon}}\left(\frac{E_0'}{E_0}\right)^2 =$$

if $\mu = \mu'$: $T = \frac{4nn'}{(n+n')^2}$; $R = \left(\frac{n-n'}{n+n'}\right)^2$

$$= \Big|_{\theta=0}^{\text{for}} = \sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}\cdot\frac{4n^2}{\left(n+\frac{\mu}{\mu'}n'\right)^2}.$$

~ fraction of incident power that got through

reflection coefficient ~ fraction of inc. power reflected. $(T+R=1)$

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \frac{E_0''\,H_0''}{E_0 H_0} = \frac{E_0''\,B_0''}{E_0 B_0} = \frac{|E_0''|^2}{|E_0|^2} = \left(\frac{n-\frac{\mu}{\mu'}n'}{n+\frac{\mu}{\mu'}n'}\right)^2$$

Electromagnetic Waves in Conductors

Maxwell equations: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\quad\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$

$$\vec{\nabla}\cdot\vec{D} = \rho = \underset{\substack{\uparrow \\ \text{assume no free charges}}}{0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

Ohm's law: $\vec{J} = \sigma\vec{E}$; $\vec{B} = \mu\vec{H}$, $\vec{D} = \varepsilon\vec{E}$

Look for plane-wave solutions:

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\ ;\quad \vec{B} = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = \underset{\substack{\parallel \\ i\omega\mu\vec{H}}}{i\omega\vec{B}} \qquad \vec{\nabla}\times\vec{H} = \underbrace{\sigma\vec{E} - i\omega\varepsilon\vec{E}}_{-i\omega\left(\varepsilon+\frac{i}{\omega}\sigma\right)}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} = i\omega\mu\,\vec{\nabla}\times\vec{H} = i\omega\mu(\sigma - i\omega\varepsilon)\vec{E}$$