$\rho(\vec{x}')$

localized charge distribution

multipole expansion:

$$\boxed{\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta,\varphi)}$$

with
$$\boxed{q_{lm} = \int d^3x' \, Y_{lm}^*(\theta',\varphi') \, r'^l \rho(\vec{x}')}$$

$$q_{00} = \frac{q}{\sqrt{4\pi}} \; ; \quad \vec{p} = \int d^3x \, \rho(\vec{x}) \, \vec{x} ; \quad Q_{ij} = \int d^3x \, \rho(\vec{x}) [3x_i x_j - r^2 \delta_{ij}] .$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[ q \frac{\vec{x}}{r^3} + \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{r^3} + \dots \right] , \quad \hat{n} = \frac{\vec{x}}{r} .$$

$W = q \, \Phi_{ext}(0) - \vec{p}\cdot\vec{E}_{ext}(0) + \dots$ electrostatic energy in external field

---

dielectrics: differential equations:

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_{free} \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$$

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

polarization (density of electric dipole moment)

LIH media: $\vec{D} = \varepsilon \vec{E} \Rightarrow$ if $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\nabla^2 \Phi = -\frac{\rho_{free}}{\varepsilon}}$

in general $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0} \vec{\nabla}\cdot(\vec{D} - \vec{P}) = \frac{\rho_{free} - \vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$

Therefore,

$$W = q\,\Phi(0) - \vec{p}\cdot\vec{E} - \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \ldots$$

## Dielectrics.

Suppose we have two types of charges:

"free charges" and "bound charges".

The potential is then the sum of potentials of free and bound charges: $\Phi = \Phi_{free} + \Phi_{bound}$

$$\Phi_{free}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho_f(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Bound charges give charge-neutral media ($e^-$ & $p$ in the atoms & molecules). The dominant multipole is dipole. (It's easy to polarize a molecule.) Potential of a point dipole $\vec{p}$ (at $\vec{x}'$) is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Def. Defining polarization $\vec{P}(\vec{x})$ as dipole moment per unit volume, we write

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

where $V$ is the region containing polarization $\vec{P}$.

As $\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} \Rightarrow$

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \vec{P}(\vec{x}')\cdot\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} = (\text{parts})$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\,\vec{\nabla}'\cdot\vec{P}(\vec{x}')$$

Finally, $\Phi(\vec{x}) = \Phi_{free}(\vec{x}) + \Phi_{bound}(\vec{x}) =$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho_f(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')\right] = \Phi(\vec{x})}$$

given $\rho_f$ & $\vec{P}(\vec{x}) \Rightarrow$ get $\Phi$.

$\Rightarrow$ There seems to be two components to total charge density: $\rho_{tot} = \rho_f - \vec{\nabla}\cdot\vec{P}$

Now, $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ true in dielectrics

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi = \frac{1}{\varepsilon_0}\left[\rho_f(\vec{x}) - \vec{\nabla}\cdot\vec{P}(\vec{x})\right]$$

as $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

**Def.** Define <u>electric displacement</u> $\vec{D}$

as $$\boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$$

Since $\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0}\left[\rho_f - \vec{\nabla} \cdot \vec{P}\right]$

$$\Rightarrow \vec{\nabla} \cdot \left[\varepsilon_0 \vec{E} + \vec{P}\right] = \rho_f \Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$$

$\vec{D}$ appears to have the meaning of electric field due to free charges.

<u>Linear Isotropic Homogeneous medium</u> (LIH):

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

linear ~ as $\vec{P} \propto \vec{E}$

homogeneous : $\chi$ is a constant, and not $\chi(\vec{x})$

isotropic : $\chi$ is independent of direction

(i.e. could be $\vec{P}_i = \varepsilon_0 \sum_j \chi_{ij} E_j$ ... )

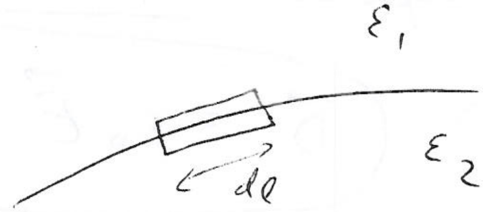
**Def** $\chi$ ~ electric susceptibility.

$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (1+\chi) \vec{E} \equiv \varepsilon \vec{E}$$

**Def** $\varepsilon = \varepsilon_0 (1+\chi)$ ~ dielectric constant. $\Rightarrow$ as $\vec{\nabla} \cdot \vec{D} = \rho_f$ $\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \rho_f / \varepsilon}$ just like vacu. $\varepsilon_0 \to \varepsilon$

(in vacuum, $\chi = 0$ and $\varepsilon = \varepsilon_0$) ; fix $\vec{D}$, take $\varepsilon \to \infty \Rightarrow \vec{E} = 0$ conductors

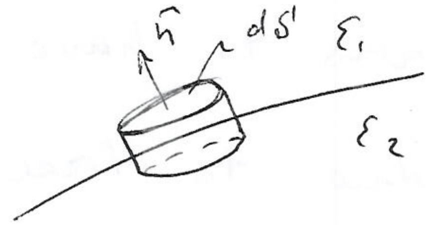
## Boundary matching




$$\vec{\nabla} \times \vec{E} = 0. \quad \Rightarrow$$

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \quad \Rightarrow \quad \left(\vec{E}_1 - \vec{E}_2\right) \cdot \hat{t}\, dl = 0$$

$$\Rightarrow \boxed{E_{1t} = E_{2t}.}$$




$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad \Rightarrow$$

$$\int_V \vec{\nabla} \cdot \vec{D}\, d^3x = \int \rho_f\, d^3x = \sigma_f \, dS$$

$$\Rightarrow \boxed{D_{1n} - D_{2n} = \sigma_f}$$

$\sigma_f$ ~ surface free charge density.

As we showed before, $E_{1n} - E_{2n} = \dfrac{\sigma}{\varepsilon_0}$

where $\sigma = \sigma_{free} + \sigma_{bound}$ ~ total surface charge density.

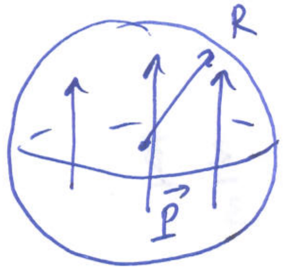
as $\vec{E} = \dfrac{1}{\varepsilon_0}\left(\vec{D} - \vec{P}\right) \Rightarrow$

$$E_{1n} - E_{2n} = \frac{1}{\varepsilon_0}\left(D_{1n} - D_{2n} - P_{1n} + P_{2n}\right) = \frac{\sigma_f}{\varepsilon_0} - \frac{P_{1n} - P_{2n}}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \left\{ P_{1n} - P_{2n} = \sigma_f - \sigma = -\sigma_b \right\}$$

Example: uniformly polarized sphere:

find $\vec{E}$, $\vec{D}$



$$\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int da' \frac{\hat{n}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|}$$

as $-\vec{\nabla}\cdot\vec{P}$ is like $\rho$ (charge density)

Spherical coordinates $\hat{n}' = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta')$

$$\vec{P} = (\underset{x}{0}, \underset{y}{0}, \underset{z}{P})$$

$$\Rightarrow \hat{n}\cdot\vec{P} = P\cos\theta' = P\cdot P_1(\cos\theta') = P\sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\varphi')$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\sum_m \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta\,\varphi)$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} P\sqrt{\frac{4\pi}{3}} \sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int d\cos\theta\, d\varphi\cdot R^2$$

$$\cdot Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, Y_{10}(\theta',\varphi') = \frac{1}{4\pi\varepsilon_0} P\sqrt{\frac{4\pi}{3}}\cdot\frac{4\pi}{3}\cdot R^2$$

$$\cdot\frac{r_<}{r_>^2} Y_{10}(\theta,\varphi) = \frac{PR^2}{\cancel{4\pi}\varepsilon_0}\frac{\cancel{4\pi}}{3}\cdot\frac{r_<}{r_>^2}\cos\theta = \frac{PR^2}{3\varepsilon_0}\frac{r_<}{r_>^2}\cos\theta$$

$$\Rightarrow \underset{r>R}{\Phi_{out} = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta}\,;\quad \underset{r<R}{\Phi_{in} = \frac{P}{3\varepsilon_0} r\cos\theta}$$

$$\vec{E}_{out} = -\vec{\nabla}\,\Phi_{out} = \frac{R^3}{3\varepsilon_0}\left[\frac{3(\hat{n}\cdot\vec{P})\,\hat{n}-\vec{P}}{r^3}\right]$$

$$\vec{E}_{in} = -\vec{\nabla}\,\Phi_{in} = -\frac{\vec{P}}{3\varepsilon_0}$$

$$\vec{D}_{out} = \varepsilon_0\vec{E}_{out}\,, \qquad \vec{D}_{in} = \varepsilon_0\vec{E}_{in}+\vec{P} = \frac{2}{3}\vec{P}.$$

as $\Phi_{out} = \dfrac{R^3}{3\varepsilon_0}\dfrac{\vec{P}\cdot\vec{r}}{r^3}$ ~ just a dipole potential

$\Phi_{in} = \dfrac{P}{3\varepsilon_0}\,z = \dfrac{\vec{P}\cdot\vec{r}}{3\varepsilon_0}$ ~ uniform $\vec{E}$ field potential