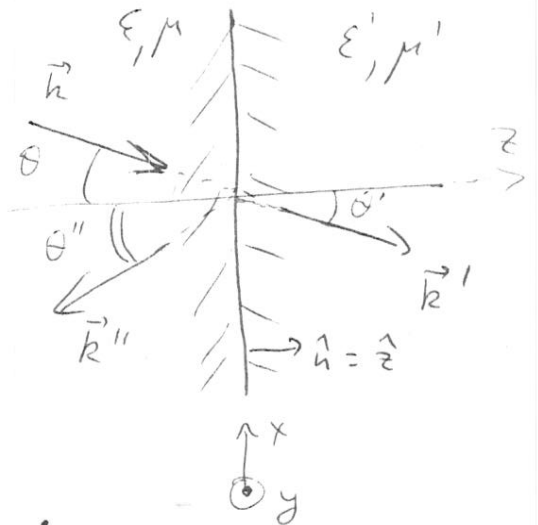


Reflection and Refraction

Incident wave:

$$\begin{cases} \vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\ \vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E} = \sqrt{\mu \epsilon} \frac{\vec{k} \times \vec{E}}{k} \end{cases}$$



Refracted wave:

$$\begin{cases} \vec{E}' = \vec{E}_0' e^{i(\vec{k}' \cdot \vec{x} - \omega' t)} \\ \vec{B}' = \sqrt{\mu' \epsilon'} \frac{\vec{k}' \times \vec{E}'}{k'} \end{cases}$$

(does not assume that all \vec{k} 's are in one plane)

Reflected wave:

$$\begin{cases} \vec{E}'' = \vec{E}_0'' e^{i(\vec{k}'' \cdot \vec{x} - \omega'' t)} \\ \vec{B}'' = \sqrt{\mu \epsilon} \frac{\vec{k}'' \times \vec{E}''}{k''} \end{cases}$$

Match boundary conditions: $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow D_n$ is cont.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow B_n$ is continuous.

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0 \Rightarrow E_t$ is continuous

as \vec{B} has no δ -function singularity at $z=0$

$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega \vec{D} \Rightarrow H_t$ is continuous (same reason, at $z=0$)

to have any boundary conditions need

$$\omega = \omega' = \omega'' \Rightarrow k = k'' = \sqrt{\mu \epsilon} \omega, \quad k' = \sqrt{\mu' \epsilon'} \omega$$

spatial phase factors should also be equal

at $z=0$: $\vec{k} \cdot \vec{x} \Big|_{z=0} = \vec{k}' \cdot \vec{x} \Big|_{z=0} = \vec{k}'' \cdot \vec{x} \Big|_{z=0}, \quad \forall x, y$

\Rightarrow choose $\vec{n} = (n_x, 0, n_z) \Rightarrow \vec{n} \cdot \vec{x} \Big|_{z=0} = n_x \cdot x \Rightarrow$ no y -dep (74)

\Rightarrow there should be no y -dependence in $\vec{k}' \cdot \vec{x}$ and in $\vec{k}'' \cdot \vec{x}$ as well $\Rightarrow k'_y = k''_y = 0 \Rightarrow$ all lie in the same plane

$$k \cdot \sin \theta = k' \cdot \sin \theta' = k'' \cdot \sin \theta''$$

\Rightarrow as $k = k'' \Rightarrow \theta = \theta''$ \approx angle of reflection is equal to angle of incidence!

as $k = \sqrt{\mu \epsilon} \omega$ and $k' = \sqrt{\mu' \epsilon'} \omega \Rightarrow$

$$\sqrt{\mu \epsilon} \sin \theta = \sqrt{\mu' \epsilon'} \sin \theta'. \quad \text{Remember } n = c \sqrt{\mu \epsilon}$$

(index of refraction) $\Rightarrow n \sin \theta = n' \sin \theta'$

Snell's law!

The only thing left is to find \vec{E}_0' & \vec{E}_0'' using b.c.'s:

$$D_n \text{ continuous} \Rightarrow \hat{n} \cdot [\epsilon (\vec{E}_0 + \vec{E}_0'') - \epsilon' \vec{E}_0'] = 0$$

$$B_n \text{ continuous} \Rightarrow \hat{n} \cdot [\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0'] = 0$$

(and $\omega = \omega' = \omega''$)

$$E_t \text{ continuous} \Rightarrow \hat{n} \times [\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'] = 0$$

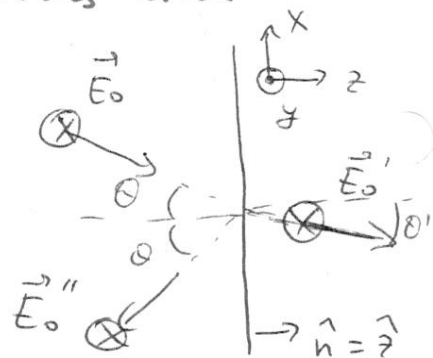
$$H_t \text{ continuous} \Rightarrow \left[\frac{1}{\mu} (\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'') - \frac{1}{\mu'} (\vec{k}' \times \vec{E}_0') \right] \times \hat{n} = 0$$

$\vec{E}_0', \vec{E}_0'' = 6$ unknowns, while there are 6 equations above

Consider 2 cases: (linear polarization)

(I) $\vec{E}_0 \perp$ to the plane of incidence

$$\vec{E}_0, \vec{E}_0', \vec{E}_0'' \parallel \hat{y}$$



3rd & 4th equations: ($\hat{u} = \hat{z}$)

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \frac{1}{\mu} (k E_0 \cos \theta - k'' E_0'' \cos \theta'') - \frac{1}{\mu'} k' E_0' \cos \theta' = 0 \end{cases}$$

as $k = k'' = \sqrt{\mu \epsilon} \omega$, $k' = \sqrt{\mu' \epsilon'} \omega \Rightarrow$ and $\theta = \theta''$

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \sqrt{\frac{\epsilon}{\mu}} (E_0 - E_0'') \cos \theta - \sqrt{\frac{\epsilon'}{\mu'}} E_0' \cos \theta' = 0 \end{cases}$$

1st eqn. $0 = 0$; 2nd eqn.: $k E_0 \sin \theta + k'' E_0'' \sin \theta'' - k' E_0' \sin \theta' = 0 \Rightarrow (E_0 + E_0'') \sin \theta - \sqrt{\frac{\mu' \epsilon'}{\mu \epsilon}} E_0' \sin \theta' = 0$

as $\sqrt{\mu \epsilon} \sin \theta = \sqrt{\mu' \epsilon'} \sin \theta'$ (Snell's law) $\Rightarrow E_0 + E_0'' - E_0' = 0$
 \Rightarrow duplicates the 3rd one.

Using Snell's law ($n \sin \theta = n' \sin \theta'$) to get rid of n' we write (work it out yourself):

$$\frac{E_0'}{E_0} = \frac{2n \cos \theta}{n \cos \theta + \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2 \theta}}$$

$$\frac{E_0''}{E_0} = \frac{n \cos \theta - \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2 \theta}}{n \cos \theta + \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2 \theta}}$$

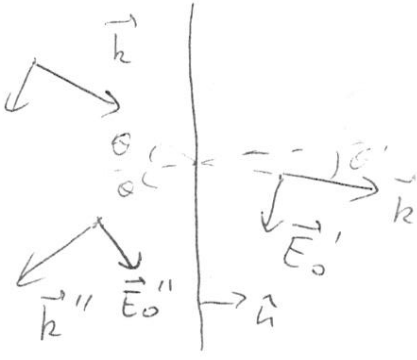
$$n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

$$n' = \sqrt{\frac{\mu' \epsilon'}{\mu_0 \epsilon_0}}$$

② $\vec{E}_0 \parallel$ plane of incidence (xz plane)

2 independent equations (3rd & 4th):

$$\begin{cases} (E_0 + E_0'') \cos \theta - E_0' \cos \theta' = 0 \\ \sqrt{\frac{\epsilon}{\mu}} (E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}} E_0' = 0 \end{cases}$$



(other two can be reduced to those)

Solve:

(using Snell's Law)

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n n' \cos \theta}{\frac{\mu}{\mu'} n'^2 \cos \theta + n \sqrt{n'^2 - n^2 \sin^2 \theta}} \\ \frac{E_0''}{E_0} &= \frac{-\frac{\mu}{\mu'} n'^2 \cos \theta + n \sqrt{n'^2 - n^2 \sin^2 \theta}}{\frac{\mu}{\mu'} n'^2 \cos \theta + n \sqrt{n'^2 - n^2 \sin^2 \theta}} \end{aligned}$$

Normal incidence: $\theta = 0 \Rightarrow$ both ① and ②

give

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'} n'}, \quad \frac{E_0''}{E_0} = \frac{n - \frac{\mu}{\mu'} n'}{n + \frac{\mu}{\mu'} n'}$$

if $\mu > \mu'$
 $n > n'$
 \Rightarrow reflected wave change sign - phase revers

Polarization by reflection: put $\mu = \mu'$ for simplicity.

①: $\frac{E_0''}{E_0} = \frac{n \cos \theta - \sqrt{n'^2 - n^2 \sin^2 \theta}}{n \cos \theta + \sqrt{n'^2 - n^2 \sin^2 \theta}}$ ← different \Rightarrow

②: $\frac{E_0''}{E_0} = \frac{-n'^2 \cos \theta + n \sqrt{n'^2 - n^2 \sin^2 \theta}}{n'^2 \cos \theta + n \sqrt{n'^2 - n^2 \sin^2 \theta}}$ $\checkmark \Rightarrow$ reflected light is polarized!

in case (I) $\frac{E_o''}{E_o}$ never vanishes (always < 0) if $n' > n$

in case (II) $\frac{E_o''}{E_o} = 0$ for $\theta_B = \tan^{-1}\left(\frac{n'}{n}\right)$ Brewster's angle

$$\left[\begin{aligned} n^2 n'^2 - n^4 \sin^2 \theta &= n'^4 \cos^2 \theta \Rightarrow \\ \Rightarrow n^2 n'^2 (1 + \tan^2 \theta) - n^4 \tan^2 \theta &= n'^4 \Rightarrow n^2 n'^2 - n^4 = n^2 (n'^2 - n^2) \tan^2 \theta \Rightarrow \tan^2 \theta = \frac{n'^2}{n^2} \end{aligned} \right]$$

\Rightarrow reflected light is polarized.

if $\theta = \theta_B \Rightarrow$ polarization is linear, \perp to the plane of incidence.

(fish in the ocean reflect light \sim squids with polarized vision can see them)

Total internal reflection: Snell's law:

$$n \sin \theta = n' \sin \theta' \leq n' \Rightarrow \theta \leq \sin^{-1}\left(\frac{n'}{n}\right) \Rightarrow$$

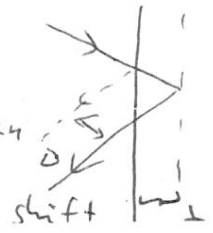
\Rightarrow for $\theta > \sin^{-1}\left(\frac{n'}{n}\right)$ get total reflection "evanescent wave"

$$\begin{aligned} k' &= \sqrt{\mu' \epsilon'} \omega \Rightarrow k'_x = -\sqrt{\mu' \epsilon'} \omega \sin \theta' = -\sqrt{\mu_0 \epsilon_0} \omega n' \sin \theta' = \\ & \quad \frac{n' \omega}{c} \qquad \qquad \qquad k'_y = 0 \qquad \qquad \qquad = -\frac{\omega}{c} n \sin \theta \\ k'_z &= \sqrt{k'^2 - k_x'^2} = \sqrt{n'^2 - n^2 \sin^2 \theta} \frac{\omega}{c} \end{aligned}$$

\Rightarrow for $\theta > \sin^{-1}\left(\frac{n'}{n}\right)$: k'_z becomes imaginary $k'_z = +i|k'_z|$

$\Rightarrow e^{i k'_z z} \sim e^{-|k'_z| z} \sim$ exponential falloff

effectively ^{the wave} gets reflected from a different surface \sim violation of geom. optics, Goos-Hänchen effect



transmission coefficient $\vec{S} = \vec{E} \times \vec{H} = \text{Re}[\vec{E} \times \vec{H}^*]$

$$T = \frac{|\vec{S}'|}{|\vec{S}|} = \frac{E_0' H_0' \cdot \frac{1}{2}}{E_0 H_0 \cdot \frac{1}{2}} = \frac{\mu}{\mu'} \frac{E_0' B_0'}{E_0 B_0} = \frac{\mu}{\mu'} \frac{\sqrt{\mu' \epsilon'} (E_0')^2}{\sqrt{\mu \epsilon} E_0^2}$$

$\uparrow \langle \cos^2 \rangle$ phase

$$T = \frac{4n n'}{(n+n')^2} \quad R = \left(\frac{n-n'}{n+n'}\right)^2$$

fraction of incident power that got through

reflection coefficient ~ fraction of inc. power reflected. (T+R=1)

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \frac{E_0'' H_0''}{E_0 H_0} = \frac{E_0'' B_0''}{E_0 B_0} = \frac{|E_0''|^2}{|E_0|^2} = \left(\frac{n - \frac{\mu}{\mu'} n'}{n + \frac{\mu}{\mu'} n'}\right)^2$$

Electromagnetic Waves in Conductors

Maxwell equations: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\vec{\nabla} \cdot \vec{D} = \rho = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$
 assume no free charges

Ohm's law: $\vec{J} = \sigma \vec{E} \quad ; \quad \vec{B} = \mu \vec{H}, \quad \vec{D} = \epsilon \vec{E}$

Look for plane-wave solutions:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad ; \quad \vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = i\omega \vec{B} \quad \vec{\nabla} \times \vec{H} = \underbrace{\sigma \vec{E} - i\omega \epsilon \vec{E}}_{-i\omega(\epsilon + \frac{i}{\omega} \sigma)}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} = i\omega \mu \vec{\nabla} \times \vec{H} = i\omega \mu (\sigma - i\omega \epsilon) \vec{E}$$