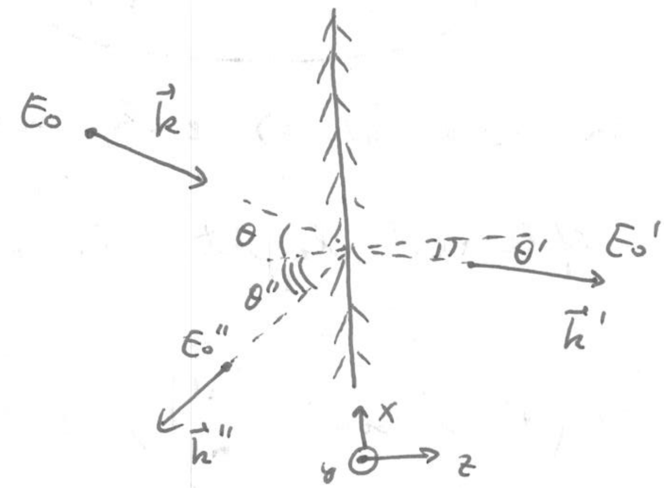
Last time | Reflection and Refraction

$$\boxed{\omega = \omega' = \omega'' \qquad k = k'' = \sqrt{\mu\varepsilon}\,\omega, \quad k' = \sqrt{\mu'\varepsilon'}\,\omega}$$



$$\boxed{\theta = \theta''}$$

$$\boxed{n \sin\theta = n' \sin\theta'}$$

(I) $\vec{E}_0 \parallel \hat{y}$ ($\vec{E}_0 \perp$ plane of incidence)

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}$$

(II) $\vec{E}_0$ in xz-plane (plane of incidence)

$$\frac{E_0'}{E_0} = \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

$$\frac{E_0''}{E_0} = \frac{-\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

$\Rightarrow$ for $\theta_B = \tan^{-1}\left(\frac{n'}{n}\right)$ (for $\mu = \mu'$) the reflected wave vanishes

in case (II) $\Rightarrow$ only have polarization $\perp$ xz-plane, reflected light is polarized

$\Rightarrow$ $\theta > \sin^{-1}\left(\frac{n'}{n}\right)$ ~ total internal reflection

$\Rightarrow \vec{E} \sim e^{-|k_z'| z}$

evanescent wave

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if $\theta = 0$ get

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'} n'} \quad , \quad \frac{E_0''}{E_0} = \frac{n - \frac{\mu}{\mu'} n'}{n + \frac{\mu}{\mu'} n'}$$

transmission coefficient $T = \frac{|\vec{S}'|}{|\vec{S}|} = \sqrt{\frac{\mu \varepsilon'}{\mu' \varepsilon}} \frac{4n^2}{\left(n + \frac{\mu}{\mu'} n'\right)^2}$ $(\theta = 0)$

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \left(\frac{n - \frac{\mu}{\mu'} n'}{n + \frac{\mu}{\mu'} n'}\right)^2 \quad (\theta = 0).$$

in case (I) $\frac{E_0''}{E_0}$ never vanishes (always <0 if $n'>n$)

in case (II) $\frac{E_0''}{E_0} = 0$ for $\boxed{\theta_B = \tan^{-1}\left(\frac{n'}{n}\right)}$ Brewster's angle

$$\left[\begin{array}{l} n^2 n'^2 - n^4 \sin^2\theta = n'^4 \cos^2\theta \Rightarrow \\ \Rightarrow n^2 n'^2 (1+\tan^2\theta) - n^4 \tan^2\theta = n'^4 \Rightarrow n^2 n'^2 - n'^4 = n^2(n^2 - n'^2)\tan^2\theta \Rightarrow \tan^2\theta = \frac{n'^2}{n^2} \end{array}\right]$$

$\Rightarrow$ reflected light is <u>polarized</u>.

if $\theta = \theta_B \Rightarrow$ polarization is linear, $\perp$ to the plane of incidence.

(fish in the ocean reflect light ~ squids with polarized vision can see them)

<u>Total internal reflection</u>: Snell's law:

$n \sin\theta = n' \sin\theta' \leq n' \Rightarrow \theta \leq \sin^{-1}\left(\frac{n'}{n}\right) \Rightarrow$

$\Rightarrow$ for $\boxed{\theta > \sin^{-1}\left(\frac{n'}{n}\right)}$ get total reflection "evanescent wave"

$k' = \sqrt{\mu'\varepsilon'}\,\omega \Rightarrow k'_x = -\sqrt{\mu'\varepsilon'}\,\omega \sin\theta' = -\sqrt{\mu_0\varepsilon_0}\,\omega\, n' \sin\theta' =$
$\quad\; \overset{\shortparallel}{n'\frac{\omega}{c}}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\frac{\omega}{c} n \sin\theta$

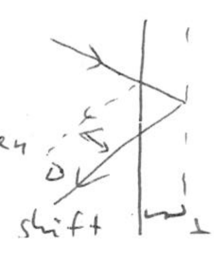
$k'_y = 0$

$k'_z = \sqrt{k'^2 - k'^2_x} = \sqrt{n'^2 - n^2\sin^2\theta}\;\frac{\omega}{c}$

$\Rightarrow$ for $\theta > \sin^{-1}\left(\frac{n'}{n}\right)$: k'_z becomes imaginary $k'_z = +i|k'_z|$

$\Rightarrow e^{ik'_z z} \sim e^{-|k'_z| z}$ ~ exponential falloff

effectively the wave gets reflected from a different surface ~ violation of geom. optics, Goos-Hänchen effect



transmission coefficient $\vec{S} = \vec{E} \times \vec{H} = Re[\vec{E} \times \vec{H}^*]$

$$T = \frac{|\vec{S}'|}{|\vec{S}|} = \frac{E_0' \cdot H_0' \cdot \frac{1}{2}}{E_0 \cdot H_0 \cdot \frac{1}{2}} = \frac{\mu}{\mu'} \frac{E_0' \cdot B_0'}{E_0 B_0} = \frac{\mu}{\mu'} \frac{\sqrt{\mu'\varepsilon'}}{\sqrt{\mu\varepsilon}} \left(\frac{E_0'}{E_0}\right)^2$$

$\uparrow \langle \cos^2 \rangle$ ~ phase

if $\mu = \mu'$ $T = \frac{4nn'}{(n+n')^2}$; $R = \left(\frac{n-n'}{n+n'}\right)^2$

$$= \Big|_{\theta=0}^{\text{for}} = \sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}} \cdot \frac{4n^2}{\left(n + \frac{\mu}{\mu'} n'\right)^2}$$

~ fraction of incident power that got through

reflection coefficient ~ fraction of inc. power reflected.
$(T + R = 1)$

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \frac{E_0'' H_0''}{E_0 H_0} = \frac{E_0'' B_0''}{E_0 B_0} = \frac{|E_0''|^2}{|E_0|^2} = \left(\frac{n - \frac{\mu}{\mu'} n'}{n + \frac{\mu}{\mu'} n'}\right)^2$$

Electromagnetic Waves in Conductors

Maxwell equations: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\vec{\nabla} \cdot \vec{D} = \rho = 0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$

$\uparrow$ assume no free charges

Ohm's law: $\vec{J} = \sigma \vec{E}$; $\vec{B} = \mu \vec{H}$, $\vec{D} = \varepsilon \vec{E}$

Look for plane-wave solutions:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} ; \quad \vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = i\omega \vec{B} \quad \vec{\nabla} \times \vec{H} = \underbrace{\sigma \vec{E} - i\omega\varepsilon \vec{E}}_{-i\omega\left(\varepsilon + \frac{i}{\omega}\sigma\right)}$$

$\quad = i\omega\mu\vec{H}$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} = i\omega\mu \vec{\nabla} \times \vec{H} = i\omega\mu(\sigma - i\omega\varepsilon)\vec{E}$$

$\Rightarrow$ we get $(\nabla^2 + k^2)\vec{E} = 0$ with $k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma$

$$\Rightarrow k = \pm\sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} \quad \boxed{\equiv k_1 + ik_2}$$

As $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$. ~ still transverse.

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$.

Good conductor: $\frac{\sigma}{\varepsilon\omega} \gg 1$

Bad conductor: $\frac{\sigma}{\varepsilon\omega} \ll 1$

Assume that $\vec{k} \parallel \hat{z}$ and $\vec{E} \parallel \hat{x}$ (linear polarization)

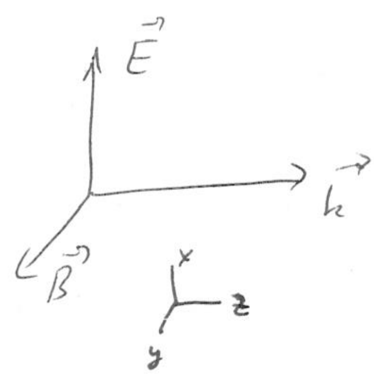
$$\vec{E} = \hat{x}E_0 e^{i(kz-\omega t)} = \hat{x}E_0 e^{ik_1 z - k_2 z - i\omega t} = \vec{E}$$

$\Rightarrow$ wavelength $\lambda = \frac{2\pi}{k_1}$

$E \sim e^{-k_2 z}$ ~ exponentially decaying

$\vec{B}$-field: $\vec{B} = -\frac{i}{\omega}\vec{\nabla}\times\vec{E} = \frac{1}{\omega}\vec{k}\times\vec{E}_0 e^{i(kz-\omega t)} \Rightarrow$

$$\vec{B} = \frac{k}{\omega}\hat{y}E_0 e^{i(kz-\omega t)}$$



$\Rightarrow \vec{B} \perp \vec{k}$, $\vec{B} \perp \vec{E}$

but: as k is complex, $\vec{B}$ & $\vec{E}$

are <u>out of phase</u>.

Time-averaged Poynting vector:

$$\langle S_z \rangle = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)_z = \frac{1}{2\mu}\mathrm{Re}\left(\frac{k^*}{\omega}|E_0|^2 e^{-2k_2 z}\right) =$$

$$= \frac{1}{2\mu} \frac{k_1 |E_0|^2}{\omega} e^{-2k_2 z} \propto e^{-z/\delta} \Rightarrow \delta = \frac{1}{2k_2}$$ "skin depth"

$$k_2 = \mathrm{Im}\left[\sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}}\right]$$

Bad conductor: $k_2 \simeq \sqrt{\mu\varepsilon}\,\cancel{\omega}\,\frac{\sigma}{2\varepsilon\cancel{\omega}} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\sigma}{2}$

$\Rightarrow \delta = \frac{1}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$ (if σ is small $\Rightarrow \delta$ is large)

Good conductor: $k_2 \simeq \sqrt{\mu\varepsilon}\,\omega\,\frac{1}{\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\sigma\mu\omega}{2}}$

$\Rightarrow \delta = \sqrt{\frac{1}{2\sigma\mu\omega}}$. (if σ is large $\Rightarrow \delta$ is small)

Frequency-dependent ε, μ, σ.

we just showed that $k = \sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} =$

$= \omega\sqrt{\mu\left(\varepsilon + \frac{i\sigma}{\omega}\right)} \Rightarrow$ if we want $k = \sqrt{\mu\varepsilon}\,\omega$

as in non-conductors, we have to, in general, assume that $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$, $\sigma = \sigma(\omega)$

and, here redefine $\varepsilon \to \varepsilon(\omega) = \varepsilon_b + \frac{i\sigma}{\omega}$ due to bound charges

"complex dielectric function" (not a constant!)

$n(\omega) = \sqrt{\frac{\mu(\omega)\,\varepsilon(\omega)}{\mu_0\,\varepsilon_0}}$ ~ "complex index of refraction"