## Last time | Electromagnetic Waves in Conductors (cont'd)

$\rho = 0, \quad \vec{J} = \sigma \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \vec{D} = \varepsilon \vec{E}$

plane wave:

$$\boxed{\left[\nabla^2 + k^2\right] \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0}$$

Helmholtz wave equation

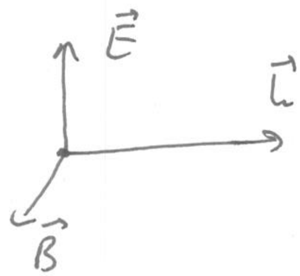
Now $k^2 = \omega^2 \mu \varepsilon + i \omega \mu \sigma$

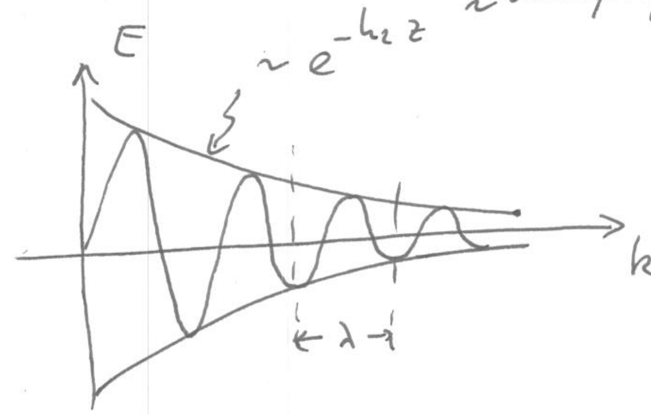
$$\Rightarrow \quad k = \pm \omega \sqrt{\mu \varepsilon} \sqrt{1 + \frac{i\sigma}{\varepsilon \omega}} \equiv k_1 + i k_2$$

$k_1, k_2 \sim$ real, $k$ is complex, $\vec{k} \parallel \hat{z}$

$$\begin{cases} \vec{E} = \hat{x} E_0 e^{-i\omega t + i k_1 z - k_2 z} \\ \vec{B} = \hat{y} \frac{k}{\omega} E_0 e^{-i\omega t + i k_1 z - k_2 z} \end{cases}$$

$\sim$ damping




$$\lambda = \frac{2\pi}{k_1}$$ wavelength

$k$ is complex $\Rightarrow$ $\vec{E}$ & $\vec{B}$ are out of phase

Good conductor: $\frac{\sigma}{\varepsilon\omega} >> 1$

$\Rightarrow k_2 \simeq \sqrt{\frac{\sigma\mu\omega}{2}} \Rightarrow$ skin depth $\delta \equiv \frac{1}{2k_2}$

( $S \sim e^{-z/\delta}$ ~ depth of penetration of light into the conductor)

$\Rightarrow \delta \simeq \frac{1}{\sqrt{2\sigma\mu\omega}}$ ~ for good conductor

$\Rightarrow$ we get $(\nabla^2 + k^2)\vec{E} = 0$ with $k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma$

$$\Rightarrow k = \pm\sqrt{\mu\varepsilon}\,\omega\sqrt{1 + \frac{i\sigma}{\varepsilon\omega}} \equiv k_1 + ik_2$$

As $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$. ~ still transverse.

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$.

Good conductor: $\frac{\sigma}{\varepsilon\omega} \gg 1$

Bad conductor: $\frac{\sigma}{\varepsilon\omega} \ll 1$

Assume that $\vec{k} \parallel \hat{z}$ and $\vec{E} \parallel \hat{x}$ (linear polarization

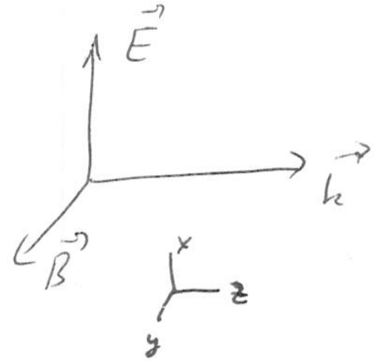
$$\vec{E} = \hat{x} E_0 e^{i(kz-\omega t)} = \hat{x} E_0 e^{ik_1 z - k_2 z - i\omega t} = \vec{E}$$

$\Rightarrow$ wavelength $\lambda = \frac{2\pi}{k_1}$

$E \sim e^{-k_2 z}$ ~ exponentially decaying

$\vec{B}$-field: $\vec{B} = -\frac{i}{\omega}\vec{\nabla}\times\vec{E} = \frac{1}{\omega}\vec{k}\times\vec{E}_0 e^{i(kz-\omega t)} \Rightarrow$

$$\vec{B} = \frac{k}{\omega}\hat{y} E_0 e^{i(kz-\omega t)}$$




$\Rightarrow \vec{B} \perp \vec{k}$, $\vec{B} \perp \vec{E}$

but: as $k$ is complex, $\vec{B}$ & $\vec{E}$

are <u>out of phase</u>.

Time-averaged Poynting vector:

$$\langle S_z \rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)_z = \frac{1}{2\mu}\mathrm{Re}\left(\frac{k^*}{\omega}|E_0|^2 e^{-2k_2 z}\right) =$$

$$= \frac{1}{2\mu}\frac{k_1|E_0|^2}{\omega}e^{-2k_2 z} \propto e^{-z/\delta} \Rightarrow \delta = \frac{1}{2k_2}$$ "skin depth"

$$k_2 = \mathrm{Im}\left[\sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}}\right]$$

Bad conductor: $k_2 \simeq \sqrt{\mu\varepsilon}\,\cancel{\omega}\,\frac{\sigma}{2\varepsilon\cancel{\omega}} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\sigma}{2}$

$\Rightarrow \delta = \frac{1}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$ (if $\sigma$ is small $\Rightarrow \delta$ is large)

Good conductor: $k_2 \simeq \sqrt{\mu\varepsilon}\,\omega\,\frac{1}{\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\sigma\mu\omega}{2}}$

$\Rightarrow \delta = \sqrt{\frac{1}{2\sigma\mu\omega}}$. (if $\sigma$ is large $\Rightarrow \delta$ is small).

## Frequency-dependent $\varepsilon, \mu, \sigma$.

we just showed that $k = \sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} =$

$= \omega\sqrt{\mu\left(\varepsilon+\frac{i\sigma}{\omega}\right)} \Rightarrow$ if we want $k = \sqrt{\mu\varepsilon}\,\omega$

as in non-conductors, we have to, in general, assume that $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$, $\sigma = \sigma(\omega)$

and, here redefine $\varepsilon \to \varepsilon(\omega) = \varepsilon_b + \frac{i\sigma}{\omega}$ due to bound charges

"complex dielectric function" (not a constant!)

$n(\omega) = \sqrt{\frac{\mu(\omega)\varepsilon(\omega)}{\mu_0\varepsilon_0}}$ ~ "complex index of refraction"

$$V_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n(\omega)} \text{ still} \Rightarrow k = \frac{n(\omega)\cdot\omega}{c}$$

(real $n(\omega)$)

<u>a simple model for $\varepsilon(\omega)$</u>: consider an electron in external electric field:

$-e\vec{v}\times\vec{B}$ is neglected: $v \sim$ small
$E, B \sim v \sim$ small
$v \times B \sim O(v^2) \sim$ tiny $\Downarrow$ drop

$$m\ddot{\vec{x}} = \vec{F} = -K\vec{x} - m\gamma\dot{\vec{x}} - e\vec{E}$$

mass; spring const; damping; constant within the "atom"

$$\omega_0^2 = \frac{K}{m} \Rightarrow m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right) = -e\vec{E}$$

if $\vec{E} = \vec{E}_0 e^{-i\omega t} \Rightarrow \vec{x} = \vec{x}_0 e^{-i\omega t} \Rightarrow$

$$\Rightarrow m\left(-\omega^2 - i\omega\gamma + \omega_0^2\right)\vec{x}_0 = -e\vec{E}_0$$

$$\Rightarrow \vec{x}_0 = \frac{e\vec{E}_0}{m(\omega^2 + i\omega\gamma - \omega_0^2)}$$

$\Rightarrow$ the amplitude of the ~~atomic~~ molecular dipole moment

$$\vec{p} = -e\vec{x}_0 = \frac{e^2\vec{E}_0}{m(\omega_0^2 - i\omega\gamma - \omega^2)}.$$

$\Rightarrow$ if there are $n$ electrons per unit volume

$$\Rightarrow \vec{P} = n\vec{p} = \frac{ne^2\,\vec{E}_0 e^{-i\omega t}}{m(\omega_0^2 - i\omega\gamma - \omega^2)} \Rightarrow \quad (\vec{E}_0 e^{-i\omega t} = \vec{E})$$

$$\Rightarrow \vec{D} = \varepsilon_0\vec{E} + \vec{P} = \left[\varepsilon_0 + \frac{ne^2}{m(\omega_0^2 - i\omega\gamma - \omega^2)}\right]\vec{E} = \varepsilon(\omega)\vec{E}$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

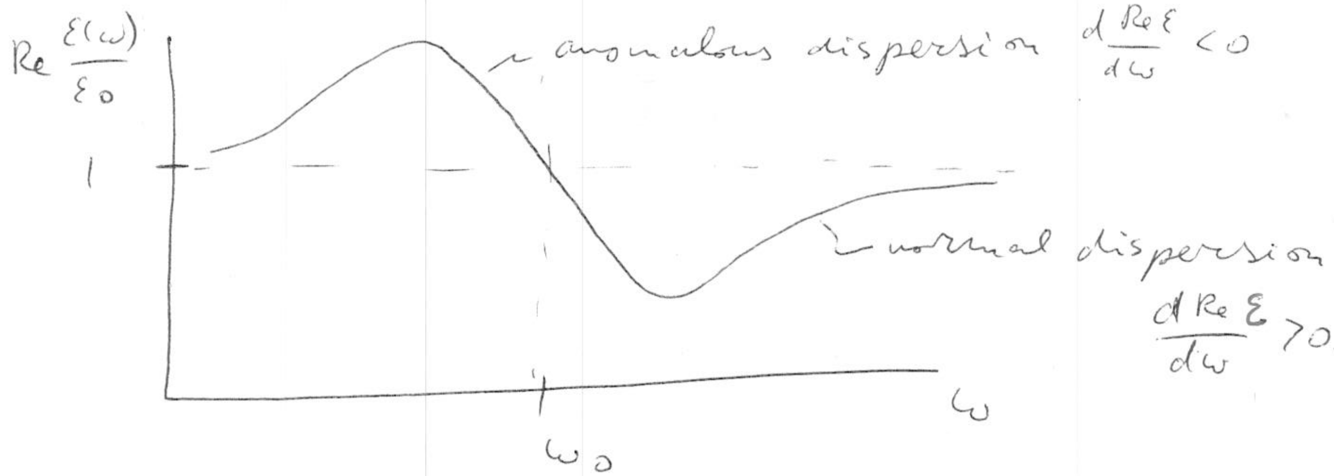
$\sim$ frequency-dependent!

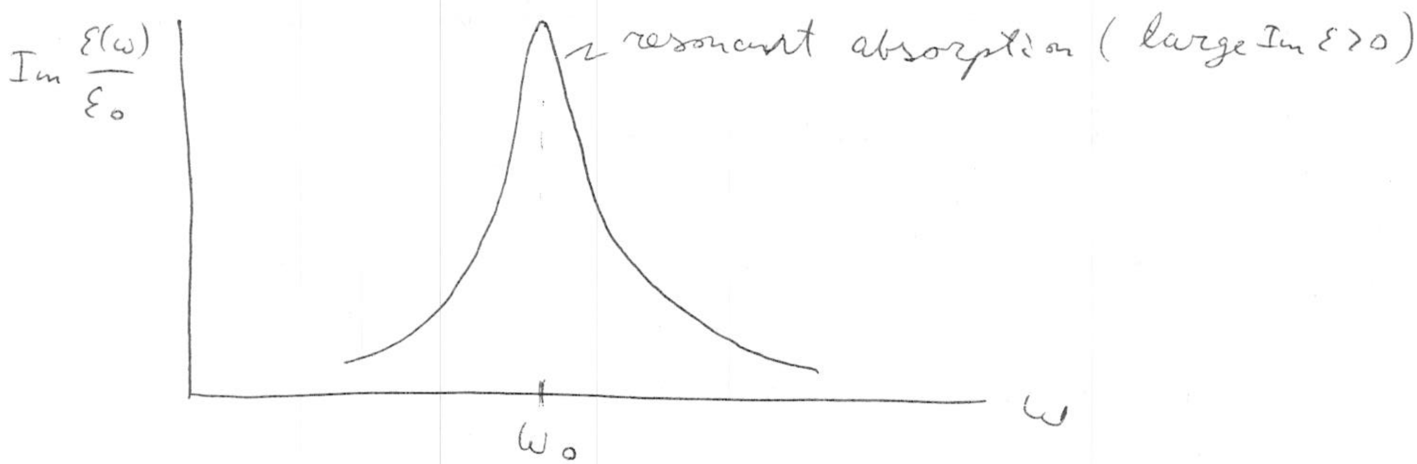
In general various electron states have different frequencies $\omega_i$, $\gamma_i$ & damping const's $\Rightarrow$ $\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{\varepsilon_0 m} \sum_i \frac{f_i}{\omega_i^2 - i\omega\gamma_i - \omega^2}$

Oscillator strength $f_i$ (# electrons/molecule with $\omega_i$ & $\gamma_i$) $\sum_i f_i = Z$

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

$$\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = \frac{ne^2}{m\varepsilon_0} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$




$$k = k_1 + ik_2 \Rightarrow |\vec{E}|^2 \sim e^{-2k_2 z} \equiv e^{-\alpha z}$$

$\alpha = 2k_2 = \frac{1}{\delta}$ = absorption coefficient (attenuation const.)

$k = \omega\sqrt{\mu(\omega)\varepsilon(\omega)}$ if $\mu(\omega) = \mu_0$ $\Rightarrow$

$$\Rightarrow k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1+\frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

$\Rightarrow k_2 \neq 0$ is $\underbrace{\ }_{\text{due to } \mathrm{Im}\,\varepsilon \neq 0 \text{, which is}}$ due to $\gamma \neq 0$ $\Rightarrow$ absorption is due to damping.

<u>Low frequency</u>: if electrons are free

$$\Rightarrow \omega_0 = 0 \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{ne^2}{m\varepsilon_0\omega(\omega + i\gamma)} =$$

on the other hand, by definition

$$= 1 + \frac{ne^2 i}{m\varepsilon_0\omega(\gamma - i\omega)} = 1 + \frac{i\sigma}{\varepsilon_0\omega} \Rightarrow$$

$$\Rightarrow \sigma_{(\omega)} = \frac{ne^2}{m}\frac{1}{\gamma - i\omega}$$

Drude model (1900) of conductivity

if $\omega \to 0 \Rightarrow \varepsilon = \mathrm{Im},\ \varepsilon \sim \frac{i}{\omega} \Rightarrow n \sim \sqrt{\frac{i}{\omega}}$
$\Rightarrow R = \left|\frac{1-n}{1+n}\right|^2 \approx 1 \Rightarrow$ metals are shiny!

<u>High frequency</u>: $(\omega \gg \omega_0,\ \omega \gg \gamma$ too$)$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{ne^2}{m\varepsilon_0\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

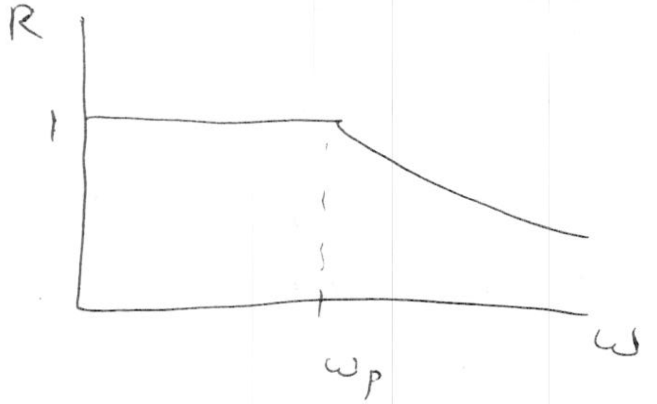
where $\omega_p^2 = \frac{ne^2}{m\varepsilon_0}$ is the <u>plasma frequency</u>

$$k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

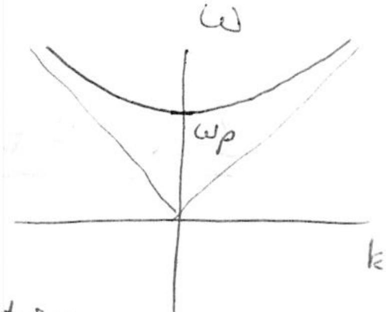
$\Rightarrow$ if $\omega < \omega_p \Rightarrow k = \frac{i}{c}\sqrt{\omega_p^2 - \omega^2}$ ~ imaginary $\Rightarrow$

$\Rightarrow$ waves do not propagate! ~ screening

Reflectivity
$$R = \left|\frac{1 - n(\omega)}{1 + n(\omega)}\right|^2 = \left|\frac{1 - \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{1 + \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}\right|^2 = \begin{cases} 1, & \omega < \omega_p \\ <1, & \omega > \omega_p \end{cases}$$




most energy is <u>reflected</u>!
(at $\omega < \omega_p$)




$$\omega^2 = c^2 k^2 + \omega_p^2 \Rightarrow \omega = \sqrt{c^2k^2 + \omega_p^2}$$

dispersion relation

cf. $E^2 = c^2k^2 + m^2c^4$ for relativistic particle of mass $m$: $\omega_p$ is like a "mass" for photons in the medium!

## <u>Kramers-Kronig Relations</u>

Is $\varepsilon(\omega)$ arbitrary? No. In fact, due to causality $\varepsilon(\omega)$ is an analytic function of $\omega$!

Suppose $\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\,\vec{E}(\vec{x},\omega)$

$$\Rightarrow \vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, D(\vec{x},\omega)\, e^{-i\omega t} =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\vec{E}(\vec{x},\omega)\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty} d\omega\, \varepsilon(\omega) e^{-i\omega t}\cdot$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\vec{E}(\vec{x},t') = \int_{-\infty}^{\infty} dt' \quad \vec{E}(\vec{x},t')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\varepsilon(\omega)\cdot$$

$$\cdot\, e^{i\omega(t'-t)} =$$