Last time | Frequency-dependent ε, μ, σ

$$\varepsilon \to \varepsilon(\omega), \quad \sigma \to \sigma(\omega), \quad \mu \to \mu(\omega).$$

Simple model: atom as a damped harmonic oscillator. Got the dielectric function (!)

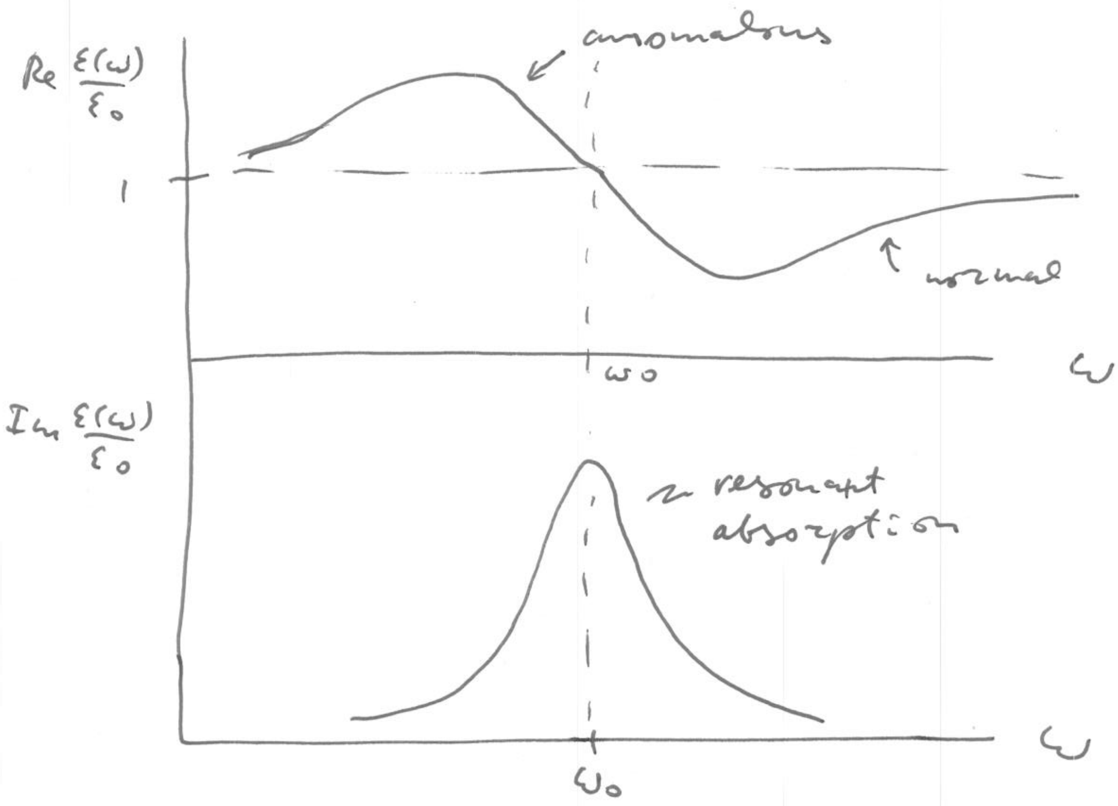
$$\boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

n = # electrons / volume

$-e$ = electron charge

ω_0 = normal frequency of the oscillator

γ = damping coefficient



$$k = \omega \sqrt{\mu(\omega)\,\varepsilon(\omega)}$$

high frequency: $\omega \gg \omega_0, \gamma$

$$\Rightarrow \quad k^2 c^2 = \omega^2 - \omega_p^2 \quad , \text{ or}$$

$$\omega^2 = k^2 c^2 + \omega_p^2$$

where $\omega_p^2 = \dfrac{ne^2}{m\varepsilon_0}$ is the plasma frequency.

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad \Rightarrow \quad \text{if } \omega < \omega_0 \Rightarrow k = i\,\mathrm{Im}\,k$$

and $e^{ikz} = e^{-|\mathrm{Im}\,k|\,z}$ ~ damped modes

$$\Rightarrow k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1+\frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

$\Rightarrow k_2 \neq 0$ is due to $\gamma \neq 0$ $\Rightarrow$ absorption is due to damping.
due to $\operatorname{Im}\varepsilon \neq 0$, which is

Low frequency: if electrons are free

$$\Rightarrow \omega_0 = 0 \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{ne^2}{m\varepsilon_0\omega(\omega+i\gamma)} =$$

on the other hand, by definition

$$= 1 + \frac{ne^2 i}{m\varepsilon_0\omega(\gamma - i\omega)} = 1 + \frac{i\sigma}{\varepsilon_0\omega} \Rightarrow$$

$$\Rightarrow \sigma_{(\omega)} = \frac{ne^2}{m}\frac{1}{\gamma - i\omega}$$

Drude model (1900) of conductivity

if $\omega \to 0 \Rightarrow \varepsilon = \operatorname{Im}$, $\varepsilon \sim \frac{i}{\omega} \Rightarrow n \sim \sqrt{\frac{i}{\omega}}$
$\Rightarrow R = \left|\frac{1-n}{1+n}\right|^2 \approx 1 \Rightarrow$ metals are shiny!

High frequency: $(\omega \gg \omega_0, \omega \gg \gamma$ too$)$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{ne^2}{m\varepsilon_0\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

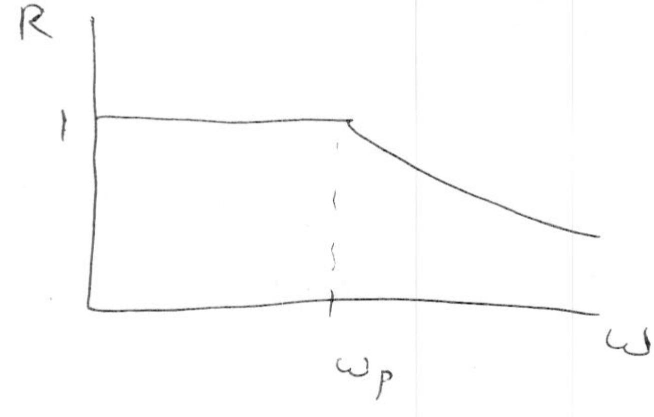
where $\omega_p^2 = \frac{ne^2}{m\varepsilon_0}$ is the plasma frequency

$$k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$$

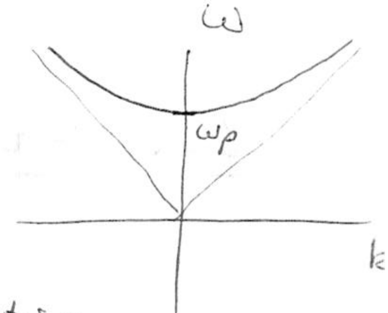
$\Rightarrow$ if $\omega < \omega_p \Rightarrow k = \frac{i}{c}\sqrt{\omega_p^2-\omega^2}$ ~ imaginary $\Rightarrow$

$\Rightarrow$ waves do not propagate! ~ screening

Reflectivity
$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 = \left|\frac{1-\sqrt{1-\frac{\omega_p^2}{\omega^2}}}{1+\sqrt{1-\frac{\omega_p^2}{\omega^2}}}\right|^2 = \begin{cases} 1, & \omega < \omega_p \\ <1, & \omega > \omega_p \end{cases}$$



most energy is <u>reflected</u>!
(at $\omega < \omega_p$)



$$\omega^2 = c^2k^2 + \omega_p^2 \Rightarrow \omega = \sqrt{c^2k^2+\omega_p^2}$$

dispersion relation

cf. $E^2 = c^2k^2 + m^2c^4$ for relativistic particle of mass m: ω_p is like a "mass" for photons in the medium!

<u>Kramers-Kronig Relations</u>

Is $\varepsilon(\omega)$ arbitrary? No. In fact, due to causality $\varepsilon(\omega)$ is an analytic function of ω!

Suppose $\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\vec{E}(\vec{x},\omega)$

$$\Rightarrow \vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, D(\vec{x},\omega)\, e^{-i\omega t} =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\vec{E}(\vec{x},\omega)\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega) e^{-i\omega t} \cdot$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\vec{E}(\vec{x},t') = \int_{-\infty}^{\infty} dt' \quad \vec{E}(\vec{x},t')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\varepsilon(\omega)\cdot$$

$$\cdot\, e^{i\omega(t'-t)} =$$

$$= \int_{-\infty}^{\infty} dt' \, \vec{E}(\vec{x},t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t'-t)} \left[\varepsilon(\omega) - \varepsilon_0 + \varepsilon_0 \right] = \Big/ \tau = t - t'$$

$$= \varepsilon_0 \vec{E}(\vec{x},t) + \varepsilon_0 \int_{-\infty}^{\infty} d\tau \, G(\tau) \, \vec{E}(\vec{x}, t-\tau)$$

such that $\vec{D}(\vec{x},t) = \varepsilon_0 \left\{ \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau \, G(\tau) \, \vec{E}(\vec{x},t-\tau) \right\}$

with $\boxed{G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right]}$

Usually $G(\tau) = 0$ for $\tau < 0 \Rightarrow$ causality : $\vec{D}(\vec{x},t)$ is affected by $\vec{E}(\vec{x},t)$ (instantaneous term) and by $\vec{E}(\vec{x},t')$ with $t' < t$ ~ **delayed action.**

Example: in a simple model above we had

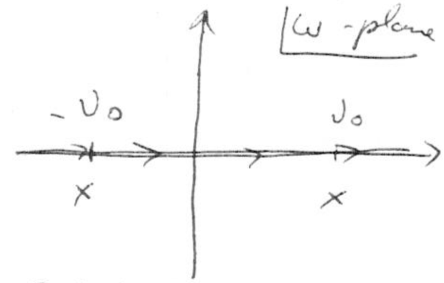
$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

$$\Rightarrow G(\tau) = \omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \frac{1}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

poles at $\omega^2 + i\omega\gamma_0 - \omega_0^2 = 0$

$$\omega_{1,2} = \frac{1}{2}\left[-i\gamma_0 \pm \sqrt{-\gamma_0^2 + 4\omega_0^2} \right] = \pm \sqrt{\underbrace{\omega_0^2 - \frac{\gamma_0^2}{4}}_{\nu_0^2}} - \frac{i\gamma_0}{2} =$$

$$= \pm \nu_0 - i \frac{\gamma_0}{2}$$



$$\Rightarrow G(\tau) = -\omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \cdot \frac{1}{\left(\omega - \nu_0 + \frac{i\gamma_0}{2}\right)\left(\omega + \nu_0 + \frac{i\gamma_0}{2}\right)} = -\omega_p^2 \cdot \Theta(\tau) \cdot (-2\pi i) \cdot \frac{1}{2\pi}$$

$$\cdot \left[\frac{1}{2\nu_0}\, e^{-i\left(\nu_0 - i\frac{\gamma_0}{2}\right)\tau} + \frac{1}{-2\nu_0}\, e^{+i\left(\nu_0 + \frac{i\gamma_0}{2}\right)\tau} \right] =$$

$$= \frac{\omega_p^2}{2\nu_0}\, \Theta(\tau) \cdot i \cdot (-2i) \sin(\nu_0\tau) \cdot e^{-\frac{\gamma_0}{2}\tau}$$

$$\Rightarrow G(\tau) = \Theta(\tau)\, \omega_p^2\, e^{-\frac{\gamma_0\tau}{2}}\, \frac{\sin(\nu_0\tau)}{\nu_0}$$

$G(\tau) \sim \Theta(\tau)$ ~ causality

$G(\tau) \sim e^{-\frac{\gamma_0}{2}\tau}$ ~ you can go back in time only so much.

Invert the expression for $G(\tau)$: first, assuming that $G(\tau) = 0$ for $\tau < 0$ write:

$$\vec{D}(\vec{x}, t) = \varepsilon_0 \left\{ \vec{E}(\vec{x}, t) + \int_0^{\infty} d\tau\, G(\tau)\, \vec{E}(\vec{x}, t - \tau) \right\}$$

$$\Rightarrow G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right] \Rightarrow$$

$$\Rightarrow \int_0^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega t} G(\tau) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \Rightarrow$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, e^{i\omega\tau} G(\tau)}$$

$\vec{E}, \vec{D}$ are real $\Rightarrow$ G is real $\Rightarrow$ $\frac{\varepsilon(-\omega)}{\varepsilon_0} = \frac{\varepsilon^*(\omega^*)}{\varepsilon_0}$

Physically reasonable $G(\tau) \to 0$ as $\tau \to \infty$ $\Rightarrow$

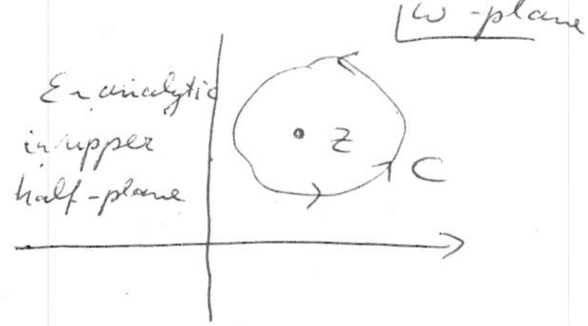
$\varepsilon(\omega)$ is analytic for $\operatorname{Im}\omega \geq 0$. (e.g. see the retarded Green fn. calculation)

(including $\operatorname{Im}\omega = 0$ ~ real axis)

$G(0) \equiv 0$ ~ continuity!

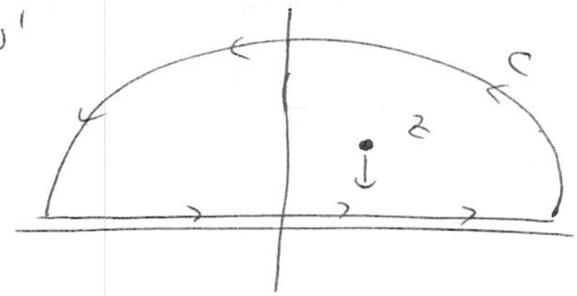
(take $\tau \to +0$)

Use Cauchy's theorem:



$$\frac{\varepsilon(z)}{\varepsilon_0} = 1 + \frac{1}{2\pi i} \oint_C \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - z}\, d\omega'$$

Distort C-contour to $\rightarrow$



and take $\operatorname{Im} z \to +0$.

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau \underbrace{e^{i\omega\tau}}_{= \frac{-i}{\omega}\frac{d}{d\tau}e^{i\omega\tau}} G(\tau) = 1 - \frac{i}{\omega}\int_0^\infty d\tau\, G(\tau) \frac{d}{d\tau} e^{i\omega\tau} =$$

$$= (\text{parts}) = 1 - \frac{i}{\omega} \underbrace{G(\tau) e^{i\omega\tau}\Big|_0^\infty}_{0} + \frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau} G'(\tau) =$$

$$= \begin{pmatrix}\text{parts}\\ \text{again}\end{pmatrix} = 1 + \underbrace{\frac{e^{i\omega\tau}}{\omega^2} G'(\tau)\Big|_0^\infty}_{= -G'(0)/\omega^2} - \frac{1}{\omega^2}\int_0^\infty d\tau\, e^{i\omega\tau} G''(\tau) = o\left(\frac{1}{\omega^2}\right)$$

$\Rightarrow$ neglect the semi-circle part of contour.

$\Rightarrow \operatorname{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) \sim \frac{1}{\omega^2}$, $\operatorname{Im}\frac{\varepsilon(\omega)}{\varepsilon_0} \sim \frac{1}{\omega^3}$ as $\omega \to \infty$.

Write $z = \omega + i\delta$, ω ~ real

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega - i\delta}$$

use
$$P\frac{1}{x} = \left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right)\frac{1}{2}$$
$$\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x)$$

as $\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + \pi i\,\delta(\omega'-\omega) \Rightarrow$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} P\left(\frac{1}{\omega'-\omega}\right)\left[\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi i} P\int_{-\infty}^{\infty} d\omega' \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\left(\varepsilon(\omega')/\varepsilon_0\right)}{\omega' - \omega} \\ \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left(\varepsilon(\omega')/\omega_0\right) - 1}{\omega' - \omega}\, d\omega' \end{aligned}}$$

Kramers - Kronig relations. '26 - '27

If you know $\mathrm{Im}\,\varepsilon(\omega)$ -> can find $\mathrm{Re}\,\varepsilon(\omega)$

& vice versa.

as $\mathrm{Re}\,\varepsilon(\omega) \sim \frac{1}{\omega^2}$ as $\omega \to \infty \Rightarrow$ define plasma frequency

as $\omega_p^2 \equiv \lim_{\omega\to\infty}\left\{\omega^2\left[1 - \frac{\varepsilon(\omega)}{\varepsilon_0}\right]\right\} \Rightarrow \omega_p^2 = \frac{2}{\pi}\int_0^{\infty} d\omega \cdot \omega \cdot \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0}$.