Last time | **Kramers-Kronig Relations** (cont'd)

$$\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\,\vec{E}(\vec{x},\omega)$$

$$\Downarrow$$

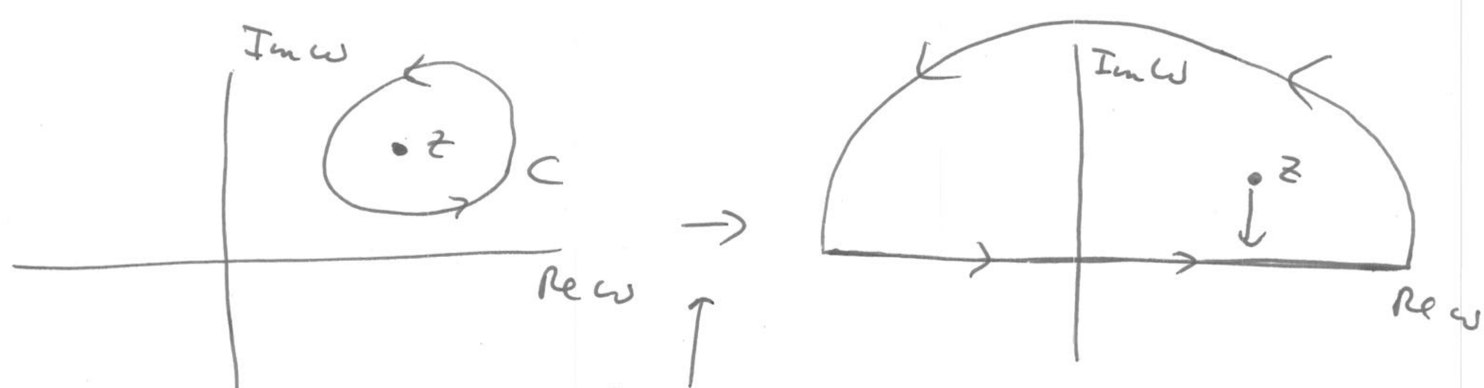
$$\boxed{\vec{D}(\vec{x},t) = \varepsilon_0\left\{\vec{E}(\vec{x},t) + \int_0^\infty d\tau\, G(\tau)\,\vec{E}(\vec{x},t-\tau)\right\}}$$

$$\boxed{G(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega\tau}\left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right]}$$

$$\Rightarrow \quad \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, e^{i\omega\tau}\, G(\tau)}$$

assume: $\left[\begin{array}{l} G(\tau) \to 0 \quad \text{as } \tau \to \infty \\ G(0) = 0 \quad (\text{continuity}) \end{array}\right.$

$\Rightarrow$ $\varepsilon(\omega)$ is analytic for $\operatorname{Im}\omega \geq 0$.



distort the contour (as $\varepsilon(\omega)$ is analytic in u.h.p.)

Got Kramers - Kronig relations:

$$\boxed{\begin{aligned} \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Im}\,\varepsilon(\omega')/\varepsilon_0}{\omega' - \omega} \\ \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega} \end{aligned}}$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, e^{i\omega\tau}\, G(\tau)}$$

$\vec{E}, \vec{D}$ are real $\Rightarrow$ G is real $\Rightarrow$ $\frac{\varepsilon(-\omega)}{\varepsilon_0} = \frac{\varepsilon^*(\omega^*)}{\varepsilon_0}$

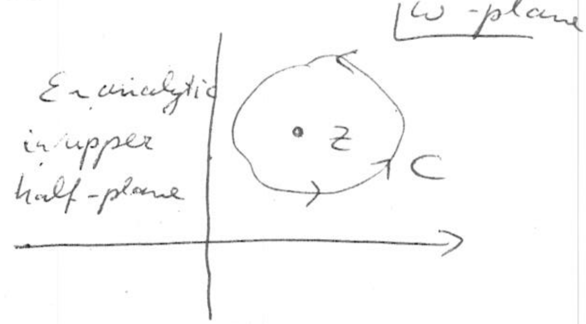
Physically reasonable $G(\tau) \to 0$ as $\tau \to \infty$ $\Rightarrow$

(e.g. see the retarded Green fn. calculation)

$\varepsilon(\omega)$ is analytic for $\mathrm{Im}\,\omega \geq 0$.
(including $\mathrm{Im}\,\omega = 0$ ~ real axis)

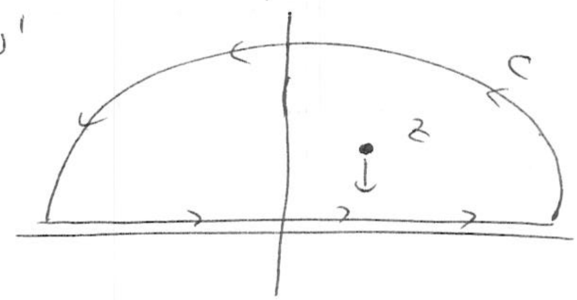
$G(0) \equiv 0$ ~ continuity!
(take $\tau \to +0$)

Use Cauchy's theorem:



$$\frac{\varepsilon(z)}{\varepsilon_0} = 1 + \frac{1}{2\pi i} \oint_C \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - z}\, d\omega'$$

Distort C-contour to $\rightarrow$



and take $\mathrm{Im}\, z \to +0$.

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, \underbrace{e^{i\omega\tau}}_{= -\frac{i}{\omega}\frac{d}{d\tau} e^{i\omega\tau}} G(\tau) = 1 - \frac{i}{\omega}\int_0^\infty d\tau\, G(\tau) \frac{d}{d\tau} e^{i\omega\tau} =$$

$$= (\text{parts}) = 1 - \frac{i}{\omega} \underbrace{G(\tau) e^{i\omega\tau}\Big|_0^\infty}_{0} + \frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau} G'(\tau) =$$

$$= \begin{pmatrix}\text{parts}\\ \text{again}\end{pmatrix} = 1 + \underbrace{\frac{e^{i\omega\tau}}{\omega^2} G'(\tau)\Big|_0^\infty}_{= -G'(0)/\omega^2} - \frac{1}{\omega^2}\int_0^\infty d\tau\, e^{i\omega\tau} G''(\tau) = o\left(\frac{1}{\omega^2}\right)$$

$\Rightarrow$ neglect the semi-circle part of contour.

$\Rightarrow \mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) \sim \frac{1}{\omega^2}$, $\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} \sim \frac{1}{\omega^3}$. as $\omega \to \infty$.

Write $z = \omega + i\delta$, $\omega \sim$ real

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega - i\delta}$$

use

$$P\frac{1}{x} = \left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right)!$$

$$\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x)$$

as $\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + \pi i\,\delta(\omega'-\omega) \Rightarrow$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} P\left(\frac{1}{\omega'-\omega}\right)\left[\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi i} P\int_{-\infty}^{\infty} d\omega' \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\left(\varepsilon(\omega')/\varepsilon_0\right)}{\omega'-\omega} \\ \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left(\varepsilon(\omega')/\omega_0\right) - 1}{\omega'-\omega}\, d\omega' \end{aligned}}$$

Kramers - Kronig relations. '26 - '27

If you know $\mathrm{Im}\,\varepsilon(\omega) \Rightarrow$ can find $\mathrm{Re}\,\varepsilon(\omega)$

& vice versa.

as $\mathrm{Re}\,\varepsilon(\omega) \sim \frac{1}{\omega^2}$ as $\omega \to \infty \Rightarrow$ define plasma frequency

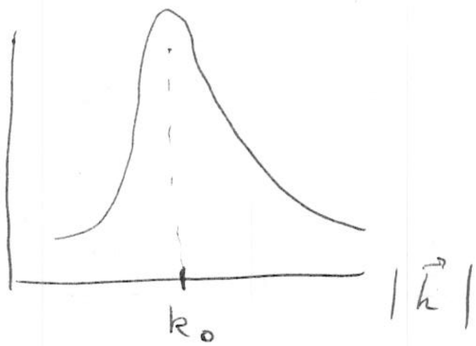
as $\omega_p^2 \equiv \lim_{\omega\to\infty}\left\{\omega^2\left[1 - \frac{\varepsilon(\omega)}{\varepsilon_0}\right]\right\} \Rightarrow \omega_p^2 = \frac{2}{\pi}\int_0^{\infty} d\omega\cdot\omega\cdot\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0}.$

Group and Phase Velocities

Consider a wave packet:

$$\vec{E}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_0(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

with $E_0(\vec{k})$



$$\Rightarrow \omega(\vec{k}) \approx \omega_0 + (\vec{k}-\vec{k}_0)\cdot(\vec{\nabla}_k\,\omega)\Big|_{\vec{k}=\vec{k}_0}$$

$$\Rightarrow \vec{k}\cdot\vec{x} - \omega t = \vec{k}\cdot\vec{x} - \omega_0 t - t(\vec{k}-\vec{k}_0)\cdot(\vec{\nabla}_k\,\omega)\Big|_{\vec{k}=\vec{k}_0} =$$

$$= \vec{k}\cdot\left(\vec{x} - t(\vec{\nabla}_k\,\omega)\Big|_{\vec{k}=\vec{k}_0}\right) \underbrace{- \omega_0 t + t\,\vec{k}_0\cdot(\vec{\nabla}_k\,\omega)\Big|_{\vec{k}=\vec{k}_0}}_{\text{overall factor (phase)}}$$

$$\Rightarrow \boxed{V_g = \vec{\nabla}_k\,\omega\Big|_{\vec{k}=\vec{k}_0}}$$ group velocity ~

as $\vec{E}(\vec{x},t) = \underbrace{e^{-i\omega_0 t + i t \vec{k}_0\cdot\vec{\nabla}_{k_0}\omega}}_{\text{overall phase}}\, \underbrace{\vec{E}(\vec{x} - t\vec{\nabla}_{k_0}\omega, 0)}_{\text{traveling wave}}$

~ speed of the wave packet

cf. $V_{ph} = \dfrac{\omega}{k}$ ~ phase velocity.

$\Rightarrow$ energy is transferred with V_g, not V_{ph}

Example: $\dfrac{\varepsilon}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2} \Rightarrow k = \omega\sqrt{\varepsilon\mu_0} = \omega\sqrt{\varepsilon_0\mu_0}\,.$

$$\sqrt{1 - \omega_p^2/\omega^2} = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \Rightarrow V_g = \frac{d\omega}{dk}\Big|_{k=k_0} =$$

$$= \frac{c}{d\sqrt{\omega^2-\omega_p^2}/d\omega} = c\,\frac{\sqrt{\omega^2-\omega_p^2}}{\omega} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}} < c$$

$(\omega > \omega_p)$

$$V_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}} > c$$ violation of relativity?

No, nothing really moves with V_{ph}, all physical quantities move with V_g!

Waveguides and Resonant Cavities.

Waves propagating in confined spaces:

cavity ~ confined in all directions

waveguide ~ confined in all but one direction ~ extended object

Consider a perfect conductor: $\sigma \to \infty$.

(skin depth $\delta \sim \frac{1}{\sqrt{\sigma}} \to 0$).

$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ and $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = 0$ inside

(otherwise $\vec{J}$ infinite)

$\vec{E} = 0 \Rightarrow \rho = 0$ inside $\Rightarrow$ can only have surface density Σ $\Rightarrow$ $\hat{n}\cdot\vec{D} = \Sigma$

as $\vec{E} = 0$ inside $\Rightarrow$ $\hat{n}\times\vec{E} = 0$

(boundary conditions)