Last time | Phase and Group Velocities

$v_{phase} = \frac{\omega}{k}$ ~ motion of wave crest

$v_{group} = \frac{d\omega}{dk}$ ~ motion of a wave packet

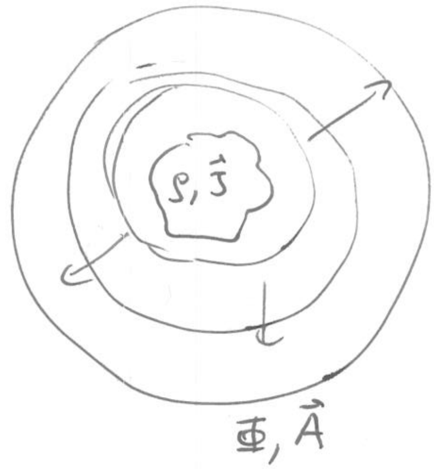
v_{phase} can be $> c$

v_{group} is always $\leq c$.

Radiation

Radiation by Harmonically Oscillating Sources

Earlier this quarter we derived Maxwell equations in Lorentz gauge:



$$\begin{cases} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\varepsilon_0} \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J} \end{cases}$$

and solved them

$$\begin{cases} \Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\, \rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\, \vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \end{cases}$$

Suppose we have harmonically oscillating localized source:

$$\begin{cases} \rho(\vec{x}, t) = \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x}, t) = \vec{J}(\vec{x})\, e^{-i\omega t} \end{cases}$$

a single frequency ω.

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\, \vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}, \quad k = \frac{\omega}{c}$$

($e^{-i\omega t}$ is understood) $\quad \vec{A}(\vec{x}, t) = \vec{A}(\vec{x})\, e^{-i\omega t}$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}$$

to find $\vec{E}$ use Ampere's law

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} =$$

$$= -i\omega\varepsilon_0 \vec{E} \quad \text{(outside the source)}$$

$$\Rightarrow \quad \vec{E} = \frac{i}{\omega \varepsilon_0} \vec{\nabla} \times \vec{H}$$

If d is source's size, $\lambda = \frac{2\pi}{k}$ is the wave length, and $d \ll \lambda, r$

then one distinguishes 3 regions:

(i) Near (static) zone $\quad d \ll r \ll \lambda$

(ii) Intermediate zone $\quad d \ll r \sim \lambda$

(iii) Far (radiation) zone $\quad d \ll \lambda \ll r$.

(i) Near zone: $e^{ik|\vec{x}-\vec{x}'|} \approx e^{i2\pi\frac{r}{\lambda}} \approx 1$

as $r \ll \lambda \Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ – just like in statics (times $e^{-i\omega t}$)

(ii) Int. zone – will discuss later.

(iii) Far zone: $\frac{r}{\lambda} \gg 1 \Rightarrow |\vec{x}-\vec{x}'| \approx r - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} =$

$$= r - \hat{n}\cdot\vec{x}' \; , \quad \hat{n} = \frac{\vec{x}}{|\vec{x}|} \; ; \quad \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r}$$

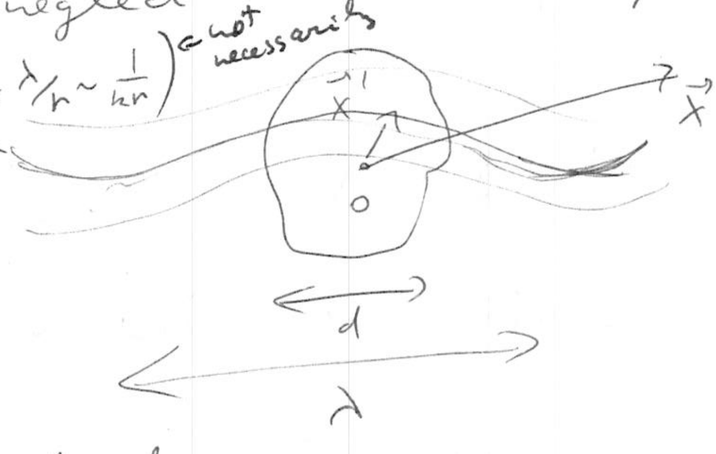
$$\Rightarrow \vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi}\,\frac{e^{ik\cdot r}}{r}\int d^3x'\,\vec{J}(\vec{x}')\,e^{-ik\hat{n}\cdot\vec{x}'}$$

(as $k\cdot r \gg 1$ ~ keep the exponent ; $k\hat{n}\cdot\vec{x}' \sim \frac{d}{\lambda}$

$$\frac{\hat{n}\cdot\vec{x}'}{r} \sim \frac{d}{r} \Rightarrow \underbrace{\frac{\hat{n}\cdot\vec{x}'}{r}}_{\text{neglect}} \sim \frac{d}{r} \ll \frac{d}{\lambda} \sim \underbrace{k\hat{n}\cdot\vec{x}'}_{\text{keep}} \quad)$$

we keep powers of d/λ $\left(\& \lambda/r \sim \frac{1}{kr}\right)$ ← not necessarily

• neglect powers of d/r.



Expanding the exponent get

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{e^{ik\cdot r}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int d^3x'\,\vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')^n}$$

~~Take n=0 term~~

<u>Monopole</u> : $\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\left(\vec{x}', t-\frac{|\vec{x}-\vec{x}'|}{c}\right) =$

$\simeq \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r}\cdot q\left(t-\frac{r}{c}\right) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r}$ ~ static $\Rightarrow$

$\Rightarrow$ can't oscillate as $\sim e^{-i\omega t}$ (or in any other way) $\Rightarrow$ no monopole term for radiation fields.

$$\left(\underset{\text{monopole}}{\Phi} \sim \int d^3x'\,\rho(\vec{x}') \sim \int_V d^3x'\,\vec{\nabla}'\cdot\vec{J}(\vec{x}') = \oint_S da'\,\hat{n}\cdot\vec{J} = \underset{\uparrow\,\text{localized}}{0}\right)$$

Electric Dipole Radiation.

take $n=0$ term in $\vec{A}$ from the far zone:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}')$$

Now, $\int d^3x' \, J_i = \int d^3x' \, J_j \underbrace{\nabla_j' x_i'}_{=\delta_{ij}} = \Big|\text{parts}\Big| =$

$$= -\int d^3x' \, x_i' \nabla_j' J_j = -\int d^3x' \, x_i' \, \vec{\nabla}' \cdot \vec{J}$$

Use continuity relation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow$

$$\Rightarrow i\omega \rho(\vec{x}) = \vec{\nabla} \cdot \vec{J} \Rightarrow \int d^3x' \, J_i = -i\omega \int d^3x' \, x_i' \rho(\vec{x}')$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{i\mu_0 \omega}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{x}' \rho(\vec{x}')$$

$\Rightarrow$ recalling that $\boxed{\vec{p} = \int d^3x' \, \vec{x}' \rho(\vec{x}')}$ is the

electric dipole moment, we get

$$\boxed{\vec{A}(\vec{x}) = -\frac{i\mu_0 \omega}{4\pi} \vec{p} \, \frac{e^{ikr}}{r}}$$ dipole radiation.

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = + \frac{i\omega}{4\pi} \vec{p} \times \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) =$$

$$= \frac{i\omega}{4\pi}\,\vec{p}\times\hat{n}\cdot\left[ik\,\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right] \Rightarrow \text{as } \omega = ck$$

$$\hat{n} = \frac{\vec{r}}{r} = \hat{r}$$

$$\vec{H} = \frac{ck^2}{4\pi}\,\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}\left[1 - \frac{1}{ikr}\right]$$

$$\vec{E} = \frac{i}{\omega\varepsilon_0}\,\vec{\nabla}\times\vec{H} \Rightarrow \text{if } kr \gg 1 \Rightarrow$$

$$\Rightarrow \boxed{\vec{H} \simeq \frac{ck^2}{4\pi}\,\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}} \Rightarrow \vec{E} = \frac{-i}{\varepsilon_0 \cancel{c}\cancel{k}}\,\frac{\cancel{c}k^2}{4\pi}\,(\hat{n}\times\vec{p})\times\vec{\nabla}\cdot$$

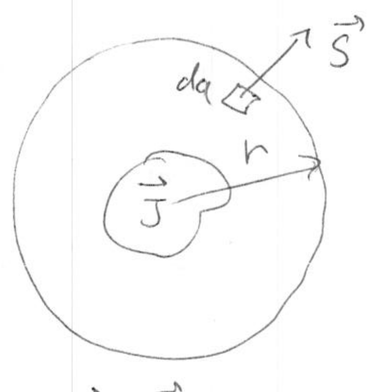
$$\frac{e^{ikr}}{r} \simeq \frac{-ik}{4\pi\varepsilon_0}\,\cancel{i}k\,(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} = \frac{1}{\varepsilon_0 c}\,\vec{H}\times\hat{n}$$

$$\Rightarrow \boxed{\vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\hat{n}}$$

$\vec{E} \perp \vec{H} \perp \hat{n}$ transverse

but only at large r.

Radiated power:



$$\frac{dP}{r^2 d\Omega} = \frac{dP}{da} = \hat{n}\cdot\vec{S}$$

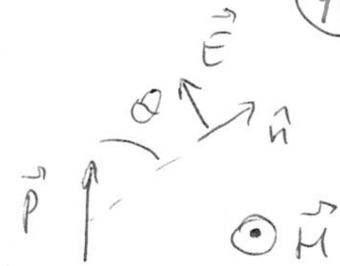
↖ unit of area

$$\Rightarrow \frac{dP}{d\Omega} = r^2\,\hat{n}\cdot\vec{S} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] =$$

↖ time-averaged!

$$= \frac{1}{2}\,r^2\sqrt{\frac{\mu_0}{\varepsilon_0}}\,|\vec{H}|^2 = \frac{1}{2}\,\cancel{r^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{c^2k^4}{(4\pi)^2}\,|\hat{n}\times\vec{p}|^2\,\frac{1}{\cancel{r^2}}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4\, |\hat{n}\times\vec{p}|^2$$

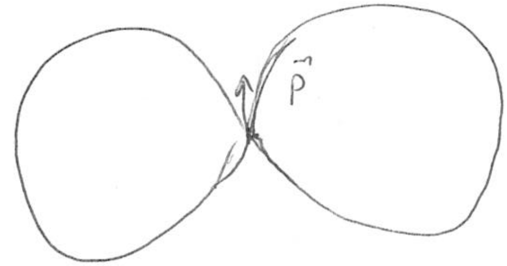


$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 \cdot \sin^2\theta.$$

Total emitted power: $P = \int d\Omega \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2.$

$$\cdot\, 2\pi \cdot \underbrace{\int_{-1}^{1} d\cos\theta\, (1-\cos^2\theta)}_{2-\frac{2}{3}=\frac{4}{3}} = \frac{c^2}{12\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 = P$$

Radiation pattern:



Examples of dipoles:

harmonically oscillating point charge on a spring

(i.e. "electron" in an "atom")