

Last time

Radiation

Radiation by Harmonically Oscillating Sources



Far (radiation) zone:

$$d \ll \lambda \ll r$$

$$\begin{cases} \rho = \rho(\vec{x}) e^{-i\omega t} \\ \vec{j} = \vec{j}(\vec{x}) e^{-i\omega t} \end{cases}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r} \int d^3x' \vec{j}(\vec{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'}, \quad k = \frac{2\pi}{\lambda}$$

keep powers of d/λ and λ/r

drop powers of $d/r \ll 1$.

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}$$

$$\vec{E} = \frac{i}{\omega \epsilon_0} \vec{\nabla} \times \vec{H}$$

Electric Dipole Radiation (cont'd)

$$\vec{A}(\vec{x}) = - \frac{i\mu_0 \omega}{4\pi} \vec{p} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$

from leading-order term in d/λ - hd expansion

$$\vec{H} \approx \frac{c k^2}{4\pi} \hat{n} \times \vec{p} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$

$$\vec{E} = \frac{1}{\epsilon_0 c} \vec{H} \times \hat{n} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H} \times \hat{n}$$

$\vec{H} \perp \vec{E} \perp \hat{n}$ at large r .

$$\vec{p} = \int d^3x \vec{x} \rho(\vec{x})$$

electric dipole moment

$$= \frac{i\omega}{4\pi} \vec{p} \times \hat{n} \cdot \left[ik \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right] \Rightarrow \text{as } \omega = ck$$

$\hat{n} = \frac{\vec{r}}{r} = \hat{r}$

$$\vec{H} = \frac{ck^2}{4\pi} \hat{n} \times \vec{p} \frac{e^{ikr}}{r} \left[1 - \frac{1}{ikr} \right]$$

$$\vec{E} = \frac{i}{\omega\epsilon_0} \vec{\nabla} \times \vec{H} \Rightarrow \text{if } kr \gg 1 \Rightarrow$$

$$\Rightarrow \vec{H} \approx \frac{ck^2}{4\pi} \hat{n} \times \vec{p} \frac{e^{ikr}}{r} \Rightarrow \vec{E} = \frac{-i}{\epsilon_0 c} \frac{ck^2}{4\pi} (\hat{n} \times \vec{p}) \times \vec{\nabla}$$

$$\frac{e^{ikr}}{r} \approx \frac{-ik}{4\pi\epsilon_0} ik (\hat{n} \times \vec{p}) \times \hat{n} \frac{e^{ikr}}{r} = \frac{1}{\epsilon_0 c} \vec{H} \times \hat{n}$$

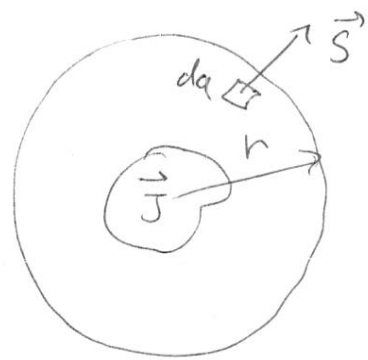
$$\Rightarrow \vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H} \times \hat{n} \quad \vec{E} \perp \vec{H} \perp \hat{n} \text{ transverse}$$

but only at large r .

Radiated power:

$$\frac{dP}{r^2 d\Omega} = \frac{dP}{d\Omega} = \hat{n} \cdot \vec{S}$$

↑ unit of area



$$\Rightarrow \frac{dP}{d\Omega} = r^2 \hat{n} \cdot \vec{S} = \frac{1}{2} \text{Re} \left[r^2 \hat{n} \cdot (\vec{E} \times \vec{H}^*) \right] =$$

$$= \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\epsilon_0}} |\vec{H}|^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{c^2 k^4}{(4\pi)^2} |\hat{n} \times \vec{p}|^2 \frac{1}{r^2}$$

↑ time-averaged!

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\hat{n} \times \vec{p}|^2$$

if all components of \vec{p} have the same phase!

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\vec{p}|^2 \sin^2 \theta$$

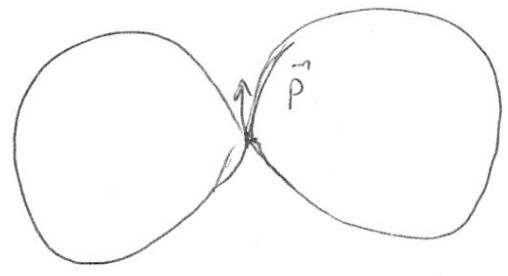


Total emitted power: $P = \int d\Omega \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\vec{p}|^2$

$$2\pi \int_{-1}^1 d\cos\theta (1 - \cos^2\theta) = \frac{c^2}{12\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\vec{p}|^2 = P$$

$$2 - \frac{2}{3} = \frac{4}{3}$$

Radiation pattern:



Examples of dipoles:

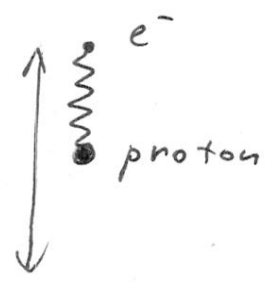
harmonically oscillating point charge on a spring
 (i.e. "electron" in an "atom")



Example | Atom as a harmonic oscillator:
(hydrogen)

the position of electron is

$$\vec{x}(t) = \hat{z} r_0 \cos(\omega t) = \hat{z} \text{Re} [r_0 e^{-i\omega t}]$$



~~$$E(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^3} \delta(x) \delta(y) \delta(z - r_0 \cos(\omega t))$$~~

~~$$(E(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^3} e^{-i\omega t})$$~~

$$\Rightarrow \vec{p} = -e \dot{\vec{x}}(t) = -e \hat{z} r_0 \omega \text{Re} e^{-i\omega t}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\vec{p}|^2 \sin^2 \theta = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 \cdot e^2 r_0^2 \omega^2 \sin^2 \theta$$

$$\cdot e^2 r_0^2 \sin^2 \theta$$

$$\Rightarrow \left(\frac{dP}{d\Omega} = \frac{1}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\omega^4}{c^2} e^2 r_0^2 \sin^2 \theta \right)$$

$$\Rightarrow P = \frac{1}{12\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\omega^4}{c^2} |\vec{p}|^2 = \left(\frac{1}{12\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\omega^4}{c^2} e^2 r_0^2 = P \right)$$

total radiated power.

~ Such classical atom would radiate energy

=> the electron would eventually fall on the proton => need quantum mechanics to save

the day!

Magnetic Dipole and Electric Quadrupole

Take the $n=1$ term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int d^3x' \vec{J}(\vec{x}') \hat{n} \cdot \vec{x}'$$

$$\vec{J}(\hat{n} \cdot \vec{x}') = \frac{1}{2} \left[(\hat{n} \cdot \vec{x}') \vec{J} + (\hat{n} \cdot \vec{J}) \vec{x}' \right] + \frac{1}{2} (\vec{x}' \times \vec{J}) \times \hat{n}$$

as $J_i x_j = \frac{1}{2} (J_i x_j + J_j x_i) + \frac{1}{2} (J_i x_j - J_j x_i)$

(a) take the 2nd term first: remember the magnetization

$$\vec{M} = \frac{1}{2} \vec{x} \times \vec{J} \Rightarrow \text{the 2nd term gives}$$

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \hat{n} \times \vec{M}(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \hat{n} \times \vec{m} \sim \text{magnetic dipole radiation.}$$

where \vec{m} is the magnetic dipole moment:

$$\vec{m} = \int d^3x' \vec{M}(\vec{x}')$$

$$\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |\hat{n} \times \vec{m}|^2$$

$$\Rightarrow \text{can find } \vec{E}, \vec{H}: \vec{H} = \frac{k^2}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m}) \times \hat{n}; \vec{E} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} k^2 \vec{m} \times \hat{n} \frac{e^{ikr}}{r}$$

(b) take the 1st term: $\frac{1}{2} \int d^3x' [n_i x'_i J_j + n_i J_i x'_j] =$

$$= \frac{1}{2} \int d^3x' \vec{J} \cdot \vec{\nabla}' (x'_j (\hat{n} \cdot \vec{x}')) = (\text{parts}) = -\frac{1}{2} \int d^3x' x'_j (\hat{n} \cdot \vec{x}')$$

$$\underbrace{\vec{\nabla}' \cdot \vec{J}}_{i\omega\rho} = -\frac{i\omega}{2} \int d^3x' x'_j (\hat{n} \cdot \vec{x}') \rho(\vec{x}') \Rightarrow$$