Last time | Magnetic Dipole and Electric Quadrupole (cont'd)

Derived expressions for the E&M radiated fields due to quadrupole oscillations:

$$\boxed{\vec{H} = -\frac{i}{24\pi}\,\omega k^2\,\frac{e^{ikr}}{r}\,\hat{n}\times\vec{Q} \qquad \vec{E} = -\frac{1}{c\varepsilon_0}\,\hat{n}\times\vec{H}}$$

$$(\vec{Q})_i = Q_{ij}\, n_j$$

$$\boxed{Q_{ij} = \int d^3x\, \rho(\vec{x})\left[3x_i x_j - \delta_{ij}\,\vec{x}^2\right]}$$ ~ electric quadrupole moment

Radiated power:

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^6}{2(24\pi)^2\varepsilon_0}\left[Q_{ij}n_j Q^*_{ik}n_k - Q_{ij}n_i n_j Q^*_{kl}n_k n_l\right]}$$

total radiated power:

$$\boxed{P = \frac{ck^6}{1440\pi\varepsilon_0}\,Q_{ij}Q^*_{ij}}$$

Alternatively,

$$\hat{n}\cdot\left[\left(\hat{n}\times(\hat{n}\times\vec{Q})\right)\times(\hat{n}\times\vec{Q}^*)\right] = \hat{n}\cdot\left[\left(\hat{n}(\hat{n}\cdot\vec{Q})-\vec{Q}\right)\times\right.$$

$$\left.\times(\hat{n}\times\vec{Q}^*)\right] = \hat{n}\cdot\left[(\hat{n}\cdot\vec{Q})\big(\cancel{\hat{n}(\hat{n}\cdot\vec{Q}^*)-\vec{Q}^*}\big)^{0} - \hat{n}|\vec{Q}|^2 + \vec{Q}^*(\hat{n}\cdot\vec{Q})\right]$$

$$= \cancel{(\hat{n}\cdot\vec{Q})^2} - |\vec{Q}|^2 + (\hat{n}\cdot\vec{Q})^2 = -\,Q_{ij}n_j Q^*_{ik}n_k +$$

$$+\ Q_{ij}n_i n_j\, Q^*_{kl}\, n_k n_l$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{ck^6}{2(24\pi)^2\varepsilon_0}\left(Q_{ij}n_j Q^*_{ik}n_k - Q_{ij}n_i n_j Q^*_{kl}n_k n_l\right)}$$

One can integrate this using $Q_{ii}=0 \Rightarrow$

$$\boxed{P = \frac{ck^6}{1440\pi\varepsilon_0}|Q_{ij}|^2.}$$

Here $|Q_{ij}|^2 \equiv Q_{ij}Q^*_{ij} = \sum_{i,j}|Q_{ij}|^2.$

<u>Example</u>: ellipsoidal oscillating charge distribution

$$\Rightarrow Q_{zz} = Q_0, \qquad Q_{xx} = Q_{yy} = -Q_0/2 \quad \text{as } Q_{ii}=0$$

$$Q_{ij}=0 \text{ if } i\neq j$$

$$\Rightarrow Q_{ij}n_j Q_{ik}n_k = +\left(\frac{Q_0}{2}\right)^2(n_x^2+n_y^2) + Q_0^2 n_z^2 =$$

$$= \frac{Q_0^2}{4}\sin^2\theta + Q_0^2\cdot\cos^2\theta$$

$$Q_{ij}n_i n_j = -\frac{Q_0}{2}(n_x^2+n_y^2) + Q_0 n_z^2 = -\frac{Q_0}{2}\sin^2\theta + Q_0\cos^2\theta$$

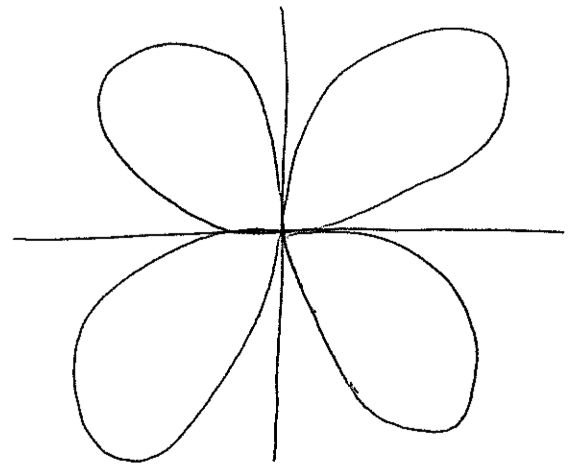
$$\Rightarrow |Q_{ij}n_i n_j|^2 = \frac{Q_0^2}{4}\sin^4\theta + Q_0^2\cos^4\theta - Q_0^2\sin^2\theta\cos^2\theta$$

$$\Rightarrow Q_{ij} n_j Q_{ik} n_k - (Q_{ij} n_i n_j)^2 = \frac{Q_0^2}{4} \sin^2\theta\cos^2\theta +$$

$$+ Q_0^2 \sin^2\theta \cos^2\theta + Q_0^2 \sin^2\theta\cos^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c k^6 \cdot 9}{2(48\pi)^2 \varepsilon_0} Q_0^2 \sin^2\theta\cos^2\theta}$$

quadrupole radiation pattern:

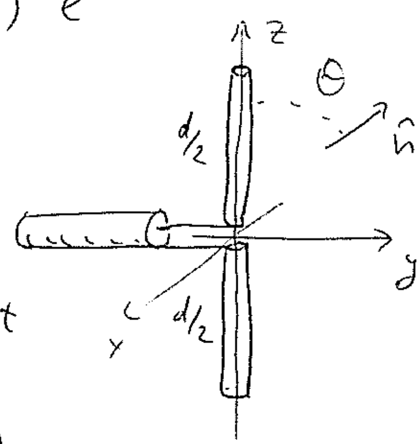


Center-Fed Linear Antenna

In some cases we do not need to expand the vector-potential in the radiation zone:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'}$$

Consider a center-fed linear antenna of length d:



$$\vec{J} = I \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\,\delta(y)\,\hat{z} \cdot e^{-i\omega t}$$

vanishes at the ends $(z = \pm d/2)$.

Plug it in:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot I\,\hat{z}\int_{-d/2}^{d/2} dz'\,\sin\left(\frac{kd}{2}-k|z'|\right)e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, I\,\hat{z}\,\frac{1}{2i}\int_{-d/2}^{d/2} dz'\left(e^{i\left(\frac{kd}{2}-k|z'|\right)} - e^{-i\left(\frac{kd}{2}-k|z'|\right)}\right)e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, I\,\hat{z}\,\frac{1}{2i}\left\{\frac{1}{-ik(1+\cos\theta)}\left(e^{-ik\frac{d}{2}\cos\theta} - e^{\frac{ikd}{2}}\right) - \right.$$

$$- \frac{1}{ik(1-\cos\theta)}\left(e^{-ik\frac{d}{2}\cos\theta} - e^{-\frac{ikd}{2}}\right) + \frac{1}{ik(1-\cos\theta)}\left(e^{i\frac{kd}{2}} - e^{\frac{ikd}{2}\cos\theta}\right)$$

$$\left. - \frac{1}{-ik(1+\cos\theta)}\left(e^{-i\frac{kd}{2}} - e^{ik\frac{d}{2}\cos\theta}\right)\right\} = \frac{\mu_0}{4\pi}\,\hat{z}\, I\,\frac{e^{ikr}}{kr}\cdot$$

$$\cdot\frac{1}{\cancel{2}i}\left\{\frac{1}{-i\cancel{k}(1+\cos\theta)}\left[\cancel{2}\cos\left(\frac{kd}{2}\cos\theta\right) - \cancel{2}\cos\left(\frac{kd}{2}\right)\right] + \right.$$

$$\left. + \frac{1}{i\cancel{k}(1-\cos\theta)}\left[\cancel{2}\cos\left(\frac{kd}{2}\right) - \cancel{2}\cos\left(\frac{kd}{2}\cos\theta\right)\right]\right\} = \frac{\mu_0}{2\pi}\,\hat{z}\, I\,\frac{e^{ikr}}{kr}\cdot$$

$$\cdot\frac{1}{\sin^2\theta}\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]$$

~ have all powers of kd included.

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{ik}{\mu_0}\,\hat{n}\times\vec{A}\ ; \quad \vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\hat{n}$$ (see dipole discussion)

acts on e^{ikr} only ~ radiation zone

$$\frac{dP}{d\Omega} = \underbrace{\frac{1}{2} \mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]}_{\vec{S}} \cdot \hat{n}\, r^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, |\vec{H}|^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot$$

$$\cdot \frac{k^2}{\mu_0^2} \cdot \sin^2\theta\, |\vec{A}|^2 = \frac{1}{2} \cancel{r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \cancel{\frac{k^2}{\mu_0^2}}\, \frac{\cancel{\mu_0^2}}{(2\pi)^2}\, I^2 \frac{1}{\cancel{k^2 r^2}} \frac{1}{\sin^4\theta} \cdot$$

$$\cdot \left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]^2 \cdot \sin^2\theta$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{I^2}{8\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2$$

radiation of center-fed antenna!

Multipole expansion in radiation zone was the expansion in $\frac{d}{\lambda} \sim kd \Rightarrow$ if $kd \ll 1$ the first term should give dipole contribution.

Expand for $kd \ll 1 \Rightarrow \frac{dP}{d\Omega} = \frac{I^2}{8\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{k^4 d^4}{64} \cdot \sin^2\theta$

$\Rightarrow \frac{dP}{d\Omega} \propto k^4 d^4 \sin^2\theta \sim$ dipole radiation