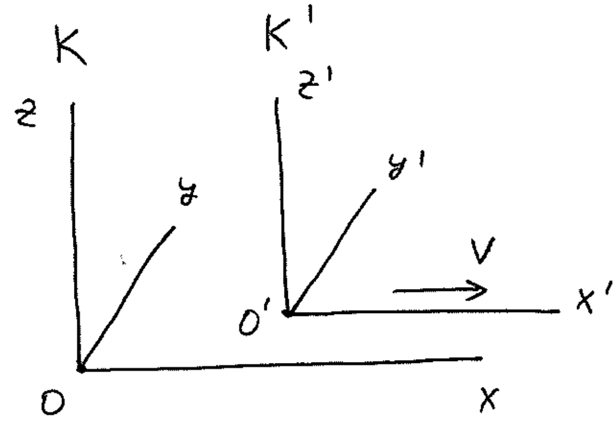
Special Theory of Relativity

Lorentz Transformations

Need to find relation between (t, x, y, z) and (t', x', y', z') in different frames.



Einstein's Postulates:

1. The laws of nature are independent of inertial frame we are in.
2. The speed of light is a finite constant and is independent of inertial frame.

Assuming that space-time is homogeneous and isotropic one can use Einstein's postulates to derive Lorentz transformations:

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

The velocity transformation:

$$u' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \Big| \, u = \frac{dx}{dt} \; = \frac{u - v}{1 - \frac{uv}{c^2}}$$

$$\Rightarrow \boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \quad \text{OR} \quad \boxed{u = \frac{u' + v}{1 + \frac{uv}{c^2}}}$$

if $u = c \Rightarrow u' = c \Rightarrow$ speed of light is c in all frames.

Def. 4-dimensional coordinates: x^μ, $\mu = 0, 1, 2, 3$

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

Def. $\beta \equiv \frac{v}{c}$, $\quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$.

$\Rightarrow$ Lorentz transformation in matrix form is

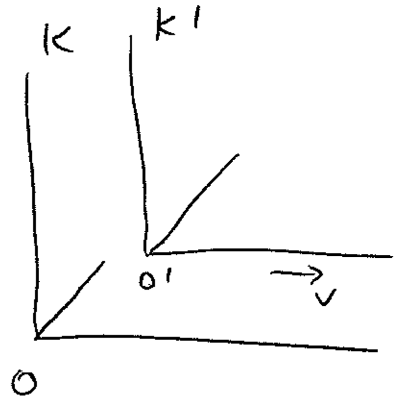
$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\equiv \Lambda^\mu{}_\nu} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$\Rightarrow$ in more compact form $\quad x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$.

The inverse Lorentz transform:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

Invariant interval: flash a light at the origin(s) at $t = t' = 0$



$\Rightarrow$ light reaches point (x, y, z) after time t such that

$$c^2t^2 - x^2 - y^2 - z^2 = 0$$

In moving frame (K') have

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = 0.$$

$\Rightarrow$ as $y' = y$, $z' = z \Rightarrow c^2t^2 - x^2 = c^2t'^2 - x'^2$

$\Rightarrow$ can explicitly check that it works:

$$c^2t'^2 - x'^2 = \frac{1}{1-\frac{v^2}{c^2}}\left[\left(ct - \frac{v}{c}x\right)^2 - (x - vt)^2\right] =$$

$$= \frac{1}{1-\frac{v^2}{c^2}}\left[c^2t^2\left(1-\frac{v^2}{c^2}\right) - x^2\left(1-\frac{v^2}{c^2}\right)\right] = c^2t^2 - x^2$$

as expected.

Quantity $S_{12}^2 = c^2 (t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$

is called the square of the interval between 2 events at times t_1 & t_2 & locations $\vec{x}_1$ & $\vec{x}_2$. S_{12}^2 is Lorentz-invariant.

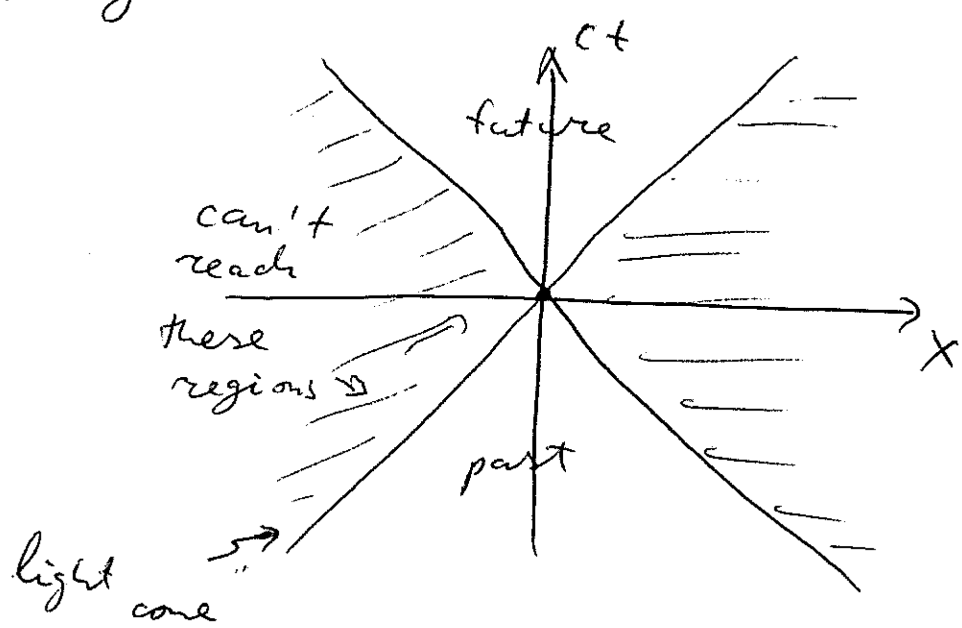
(i) $S_{12}^2 > 0 \Rightarrow$ timelike separation $\Rightarrow$ there exists a frame where $\vec{x}_1' = \vec{x}_2' \Rightarrow S_{12}^2 = c^2 (t_1'^2 - t_2'^2)$

$\Rightarrow$ the events take place at the same space point, but at diff. times

(ii) $S_{12}^2 < 0 \Rightarrow$ spacelike separation $\Rightarrow$ there exists a frame where $t_1'' = t_2'' \Rightarrow$

$\Rightarrow S_{12}^2 = -(\vec{x}_1'' - \vec{x}_2'')^2 \Rightarrow$ events take place at the same time but at different locations

(iii) $S_{12}^2 = 0 \Rightarrow$ lightlike separation



Proper time & Time Dilation.

Definition Proper time is the time in the rest frame of an object: $\boxed{d\tau \equiv \frac{ds}{c}}$.

Example: imagine a frame in which a particle moves with velocity $\vec{u}(t) \Rightarrow d\vec{x} = \vec{u}(t)\,dt$

$$\Rightarrow ds^2 = c^2 dt^2 - (d\vec{x})^2 = c^2 dt^2 - \vec{u}^2 dt^2 =$$

$$= c^2 dt^2 \left(1 - \beta^2(t)\right).$$

In the rest frame of the particle

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\beta^2(t)} \Rightarrow \boxed{\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}}.$$

alternatively

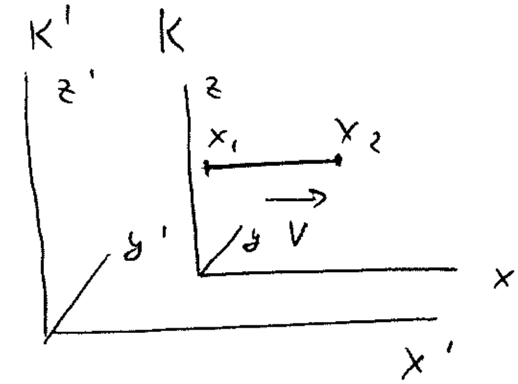
$$\boxed{t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}}} \Rightarrow \Delta t \geq \Delta\tau \text{ - time dilation.}$$

(E.g.: a photon has $\beta = 1 \Rightarrow \Delta\tau = 0 \Rightarrow$

$\Rightarrow$ the whole lifetime of the Universe is instantaneous for a photon!)

Lorentz Contraction.

Imagine a bar (rod) moving at constant velocity (see figure).



Proper length is defined as its length in the rest frame of the bar (frame K).

$$\Rightarrow x_1 = \frac{x_1' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x_2 = \frac{x_2' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta x = x_2 - x_1 = \frac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

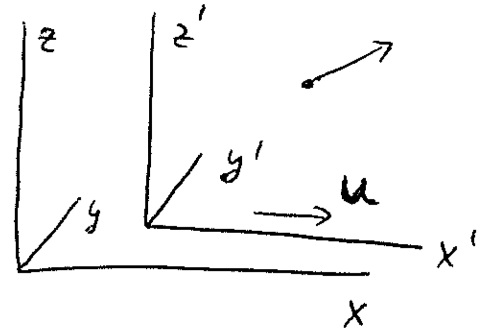
$\Rightarrow$ if l_0 is proper length, $l_0 = \Delta x$

$$\Rightarrow l = \Delta x' \Rightarrow l = l_0 \sqrt{1-\frac{v^2}{c^2}}$$

$\Rightarrow l \le l_0$ Lorentz contraction.

~ objects appear shorter in other frames.

Velocity Transformations



$$\begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ ct = (t'c + \beta x')\gamma \end{cases}$$

$$\Rightarrow V_x = \frac{dx}{dt} = \frac{dx' + u\,dt'}{dt' + \frac{\beta}{c}dx'} = \frac{V_x' + u}{1 + \frac{uV_x'}{c^2}} = V_x$$

where $V_x' = \dfrac{dx'}{dt'}$.

$$V_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\beta}{c}dx'\right)} = \frac{V_y'}{\gamma\left(1 + \frac{\beta}{c}V_x'\right)} = \frac{V_y'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}} = V_y$$

Similarly
$$V_z = \frac{V_z'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}}$$

($c \to \infty$ get Galilean
$V_x = V_x' + u$, $V_y' = V_y$, $V_z' = V_z$)

Imagine the case when $V_x = V\cos\theta$, $V_y = V\sin\theta$, $V_z = 0$

$$\Rightarrow V_z' = 0, \quad \tan\theta = \frac{V_y}{V_x} = \frac{V_y'}{\gamma(V_x' + u)} \Rightarrow$$

$\Rightarrow$ write $V_x' = V'\cos\theta'$
$V_y' = V'\sin\theta'$