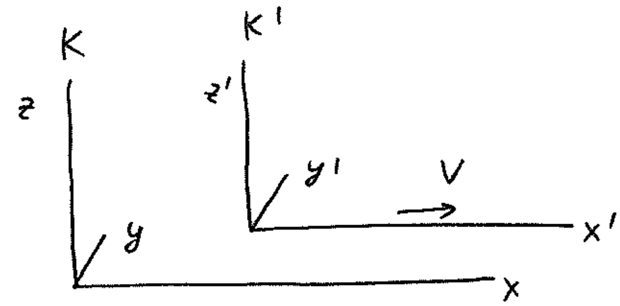
Last time

Special Theory of Relativity (cont'd)

Lorentz Transformations



$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$

$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

or $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \mu, \nu = 0, 1, 2, 3$

Def. interval $\quad ds^2 = c^2 dt^2 - d\vec{x}^2$

it is Lorentz-invariant

Proper time: $\quad d\tau = \frac{ds}{c} \quad$ (time in rest frame)

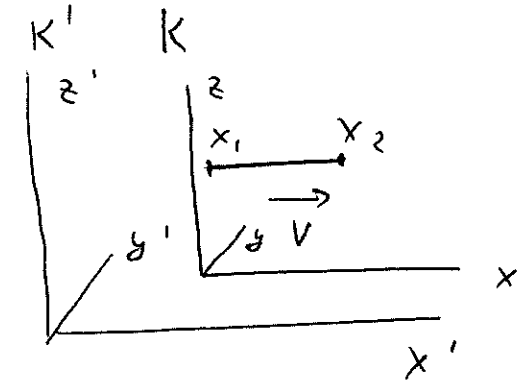
$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt \sqrt{1-\beta^2(t)} \Rightarrow \Delta\tau \leq \Delta t \quad \text{time dilation}$$

Lorentz Contraction: $\ell = \ell_0 \sqrt{1-\beta^2}$, where

ℓ_0 = proper length = length in rest frame

Lorentz Contraction.

(rod)
Imagine a bar moving at constant velocity (see figure).



Proper length is defined as its length in the rest frame of the bar (frame K).

$$\Rightarrow x_1 = \frac{x_1' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x_2 = \frac{x_2' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta x = x_2 - x_1 = \frac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\Rightarrow$ if l_0 is proper length, $l_0 = \Delta x$

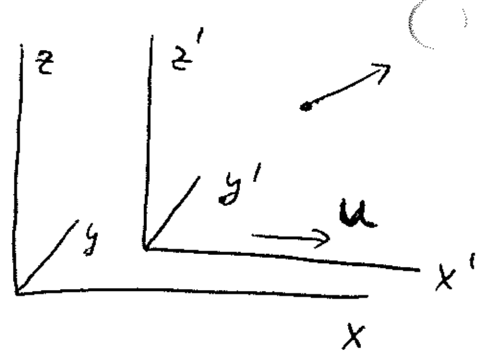
$$\Rightarrow l = \Delta x' \Rightarrow l = l_0 \sqrt{1-\frac{v^2}{c^2}}$$

$\Rightarrow l \leq l_0$ Lorentz contraction.

~ objects appear shorter in other frames.

Velocity Transformations

$$\begin{cases} x = \gamma\,(x' + u t') \\ y = y' \\ z = z' \\ ct = (t'c + \beta x')\,\gamma \end{cases}$$



$$\Rightarrow V_x = \frac{dx}{dt} = \frac{dx' + u\,dt'}{dt' + \frac{\beta}{c}dx'} = \frac{V_x' + u}{1 + \frac{uV_x'}{c^2}} = V_x$$

where $V_x' = \dfrac{dx'}{dt'}$.

$$V_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\beta}{c}dx'\right)} = \frac{V_y'}{\gamma\left(1 + \frac{\beta}{c}V_x'\right)} = \frac{V_y'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}} = V_y$$

Similarly
$$V_z = \frac{V_z'\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uV_x'}{c^2}}$$

($c \to \infty$ get Galilean $V_x = V_x' + u$, $V_y' = V_y$, $V_z' = V_z$)

Imagine the case when $V_x = V\cos\theta$, $V_y = V\sin\theta$, $V_z = 0$

$$\Rightarrow V_z' = 0\,, \quad \tan\theta = \frac{V_y}{V_x} = \frac{V_y'}{\gamma\,(V_x' + u)} \quad \Rightarrow$$

$\Rightarrow$ write $V_x' = V'\cos\theta'$

$V_y' = V'\sin\theta'$

$$\Rightarrow \tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + u)}$$

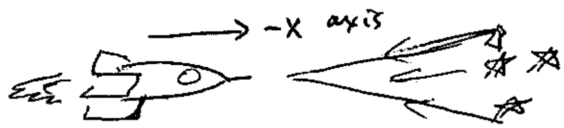
Light abberation: if the particle is a photon $\Rightarrow v = v' = c \Rightarrow \tan\theta = \dfrac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$, $\beta = \dfrac{u}{c}$, $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$$\Rightarrow v_x = \frac{v_x' + u}{1 + \frac{u v_x'}{c^2}} \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$v_y = \frac{v_y'}{\gamma\left(1 + \frac{u v_x'}{c^2}\right)} \Rightarrow \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}$$

$\Rightarrow$ if $u \to c \Rightarrow \sin\theta \to 0$, $\cos\theta \to 1 \Rightarrow$

$\Rightarrow \theta \to 0 \Rightarrow$ For UR spaceship all stars appear in the small cone ahead, at $\theta = 0$.



K ~ space ship frame, K' ~ Universe, $\theta = 0$ means the light comes along +x axis

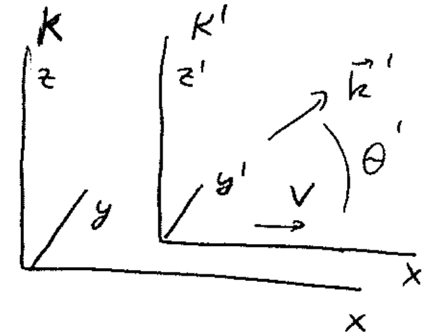
~~Four - vectors.~~

~~$x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$~~

~~(x_0, x_1, x_2, x_3) is a 4-vector.~~

Relativistic Doppler Shift.

Consider a plane wave with frequency ω & wave vector $\vec{k}$ in frame K. Assume that it has frequency ω' & wave vector $\vec{k}'$ in frame K'.



The phase of the wave is a number

$\Rightarrow$ invariant $\Rightarrow \varphi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$

$\Rightarrow$ if the boost is done in the X-direction

$\Rightarrow$ $k_y = k_y'$, $k_z = k_z'$ (as $y = y'$, $z = z'$ ~ can vary $\Rightarrow$ equate the coefficients)

$x' = \gamma(x - Vt)$, $t' = \gamma\left(t - \frac{\beta}{c}x\right)$ $\Rightarrow$

$$\Rightarrow \underline{\omega t} - k_x \cdot x = \omega' \gamma \left(\underline{t} - \frac{\beta}{c}x\right) - k_x' \cdot \gamma(x - V\underline{t})$$

$$\left.\begin{aligned} \omega &= \gamma(\omega' + k_x' V) \\ k_x &= \gamma\left(\omega'\frac{\beta}{c} + k_x'\right) \end{aligned}\right\} \Rightarrow \begin{aligned} \omega' &= \gamma(\omega - V k_x) \\ k_x' &= \gamma\left(k_x - \beta\frac{\omega}{c}\right) \end{aligned}$$

For $\vec{k}$ pointing in any random direction:

define $k^0 = \omega/c$, $k^{0\prime} = \omega'/c$ $\Rightarrow$ $\vec{\beta} \equiv \vec{V}/c$ $\Rightarrow$

$$\Rightarrow \begin{cases} k^{0\prime} = \gamma(k^0 - \vec{\beta}\cdot\vec{k}) \\ k_\parallel' = \gamma(k_\parallel - \beta k^0) \\ \vec{k}_\perp' = \vec{k}_\perp \end{cases}$$

where $k_\parallel$ is component $\parallel \vec{V}$

$\vec{k}_\perp$ are components $\perp \vec{V}$.

For an EM wave: $|\vec{k}| = k^0 = \frac{\omega}{c}$, $|\vec{k}'| = k^{0\prime} = \frac{\omega'}{c}$

$\Rightarrow$ if the angle between $\vec{k}$ and $\vec{v}$ is θ in K

and θ' in K' $\Rightarrow$ $\omega' = \gamma(\omega - \beta\,\omega\cos\theta)$ $\Rightarrow$

$$\Rightarrow \omega' = \gamma\omega(1 - \beta\cos\theta)$$ Doppler shift

$$\begin{cases} \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \beta\cdot\frac{\omega}{c}\right) \\ \frac{\omega'}{c}\sin\theta' = \frac{\omega}{c}\sin\theta \end{cases}$$

$$\Rightarrow \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$ (cf. with light abberation.)

Four - vectors.

We have seen one example: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Definition A 4-vector A^μ is a set of 4 quantities (A^0, A^1, A^2, A^3), which under Lorentz transformation transform as

$$\begin{pmatrix} A^{0\prime} \\ A^{1\prime} \\ A^{2\prime} \\ A^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

$\Rightarrow$ A^μ, $\mu = 0, 1, 2, 3$ is a <u>contravariant</u> vector if it transforms according to:

$$\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu} \quad \text{(equivalent to above)}$$

$\Rightarrow$ B_μ, $\mu = 0, \dots, 3$ is a <u>covariant</u> vector

if $$\boxed{B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu}$$

<u>Example</u>: $\frac{\partial \varphi}{\partial x^\mu}$ is a covariant vector as

$$\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu}.$$

One can define tensors by

$$A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \cdot A^\alpha B^\beta$$ $\Rightarrow$ rank two contravariant tensor would be $$C'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} C^{\alpha\beta}$$, etc.