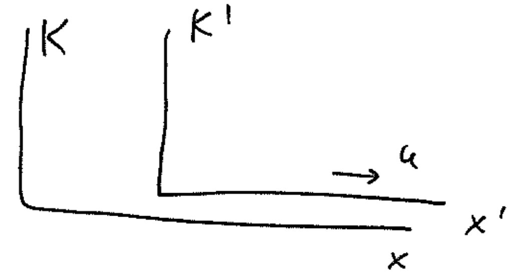
Last time | Velocity Transformations



$$\boxed{\begin{aligned} V_x &= \frac{V_x' + u}{1 + \frac{u V_x'}{c^2}} \\ V_y &= \gamma \frac{V_y'}{1 + \frac{u V_x'}{c^2}} \\ V_z &= \gamma \frac{V_z'}{1 + \frac{u V_x'}{c^2}} \end{aligned}}$$

$$\tan\theta = \frac{V'\sin\theta'}{\gamma(V'\cos\theta' + u)}$$

Relativistic Doppler Shift

plane monochromatic wave with frequency ω and wave vector $\vec{k}$ in one frame (K) and ω' & $\vec{k}'$ in K'.

$k^0 \equiv \omega/c$, $k'^0 = \omega'/c$, $\vec{\beta} = \vec{v}/c$ ~ boost velocity/c

$$\Rightarrow \begin{cases} k'^0 = \gamma(k^0 - \vec{\beta}\cdot\vec{k}) \\ k'_\parallel = \gamma(k_\parallel - \beta k^0) \\ \vec{k}'_\perp = \vec{k}_\perp \end{cases}$$

$$\Rightarrow \omega' = \gamma\omega(1 - \beta\cos\theta)$$ Doppler shift

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$ light abberation

For an EM wave: $|\vec{k}| = k^0 = \frac{\omega}{c}$, $|\vec{k}'| = k^{0\prime} = \frac{\omega'}{c}$

$\Rightarrow$ if the angle between $\vec{k}$ and $\vec{v}$ is θ in K and θ' in K' $\Rightarrow$ $\omega' = \gamma(\omega - \beta\,\omega\cos\theta)$ $\Rightarrow$

$$\Rightarrow \quad \omega' = \gamma\omega(1 - \beta\cos\theta) \quad \text{Doppler shift}$$

$$\begin{cases} \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \beta\cdot\frac{\omega}{c}\right) \\ \frac{\omega'}{c}\sin\theta' = \frac{\omega}{c}\sin\theta \end{cases}$$

$$\Rightarrow \quad \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

(cf. with light abberation.)

Four - vectors.

We have seen one example: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

A general Lorentz transformation can be

- a boost along $x, y,$ or z axis ($\forall$ angle θ)
- a rotation around the $x, y,$ or z axis ($\forall\ \beta$)

Definition A 4-vector A^μ is a set of 4 quantities (A^0, A^1, A^2, A^3), which under Lorentz transformation transform as $\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu}$ (just like x^μ)

Example boost along the x-axis:

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

$\mu, \nu = 0, 1, 2, 3$
summation over ν is implied.

$\Rightarrow$ A^μ, $\mu = 0, 1, 2, 3$ is a contravariant vector if it transforms according to:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$ (equivalent to above)

$\Rightarrow$ B_μ, $\mu = 0, \ldots, 3$ is a covariant vector if

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu$$

Example: $\frac{\partial \varphi}{\partial x^\mu}$ is a covariant vector as

$$\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu}.$$

One can define tensors by

$$A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \cdot A^\alpha B^\beta$$ $\Rightarrow$ rank two contravariant tensor would be $C'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} C^{\alpha\beta}$, etc.

In general $T^{\nu_1 \ldots \nu_m}_{\mu_1 \ldots \mu_n}$ transforms as $A_{\mu_1} \ldots A_{\mu_n} B^{\nu_1} \ldots B^{\nu_m}$.

Definition Scalar (inner) product of 2 vectors is defined by $A_\mu \cdot B^\mu$ (summation assumed)

Let's prove that it's Lorentz-invariant:

$$A'_\mu \cdot B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta \cdot$$

$$\cdot A_\alpha B^\beta = A_\alpha B^\alpha \; . \quad \text{Q.E.D.}$$

The interval is a scalar: (it's Lorentz-invariant)

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Define the <u>metric tensor</u> by

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

(Minkowski space) Cartesian coordinates

Note that $dx_\mu dx^\mu$ is also a Lorentz-scalar.

Identifying $dx_\mu = g_{\mu\nu} dx^\nu$ we see that

$g_{\mu\nu}$ lowers indices of 4-vectors, tensors, etc.

Example: $x^\mu = (ct, \vec{x})$, $x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})$.

(Def.) $g^{\mu\nu}$ is an inverse matrix for $g_{\mu\nu}$: $\boxed{g^{\mu\nu} g_{\rho\nu} = \delta^\mu_\nu}$

where $\delta^\mu_\nu = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$.

$$A_\mu = g_{\mu\nu} A^\nu \underset{\text{invert } g_{\mu\nu} \text{ matrix}}{\Longrightarrow} A^\mu = g^{\mu\nu} A_\nu$$

(e.g. $x_\mu = g_{\mu\nu} x^\nu, \ldots$)

$\Rightarrow$ $g^{\mu\nu}$ raises indices of 4-vetors, tensors, etc.

$$\delta^\mu{}_\nu = g^{\mu\alpha} \cdot g_{\alpha\nu} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix} = g_{\mu\nu}$$

only for Minkowski space!

indeed, as $A_\mu B^\mu = \delta^\mu{}_\nu \cdot A_\mu B^\nu = g^\mu{}_\nu A_\mu B^\nu$.

(Define) an abbreviated notation: $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$\Rightarrow$ $\partial_\mu \varphi$ is a covariant vector

$\partial^\mu \varphi$ is a contravariant vector (check!)

$$\left(\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu} \right)$$

$\partial_\mu A^\mu$ is Lorentz - invariant

4d Laplace operator $\dfrac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu$ $\left(= \frac{\partial}{\partial x^0}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x_2}\right)$ is

(d' Alambertian)

also Lorentz - invariant. (can you prove this?)

4 - velocity

Let's define a 4-vector for velocity:

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow V^\mu \overset{?}{=} \frac{dx^\mu}{dt} \; ?$$

But: time is not a scalar!

$$\frac{dx^\mu}{dt} \sim \frac{dx^\mu}{dx^0} \sim \text{not a Lorentz-vector.}$$

$\Rightarrow$ try proper time $d\tau = \frac{ds}{c} \Rightarrow$ $u^\mu \equiv \frac{dx^\mu}{d\tau}$ 4-velocity

$$\text{as } d\tau = \frac{dt}{\gamma} \Rightarrow u^0 = \frac{c\,dt}{dt/\gamma} = c\gamma$$

$$\vec{u} = \frac{d\vec{x}}{dt}\cdot\gamma = \gamma\cdot\vec{v} \Rightarrow \boxed{u^\mu = \gamma(c, \vec{v})}.$$

Note $u_\mu u^\mu = c^2$.

Boost in terms of rapidity.

$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$\Rightarrow$ Define rapidity η

by $\beta \equiv \tanh\eta = \dfrac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}$

Define light-cone coordinates $A^+ = \dfrac{A^0 + A^1}{\sqrt{2}}$

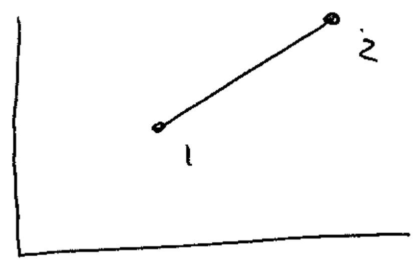
$A^- = \dfrac{A^0 - A^1}{\sqrt{2}}$

$\Rightarrow$ then

$$A_\mu A^\mu = 2A^+A^- - (A^2)^2 - (A^3)^2$$

Relativistic Mechanics.

Consider a free particle (moving along a straight line). We need to construct a Lorentz-invariant action for such particle. It's characterized by a 4-vector x^μ $\Rightarrow$ the only Lorentz-invariant is the interval $\Rightarrow$



$$\Rightarrow \int_1^2 ds \qquad \left(\text{can't have } \int_1^2 (ds)^2 \sim \text{still infinitesimal}\right)$$

$\Rightarrow$ write action S as $\mathcal{S} = -A \cdot \int_1^2 ds$

As $ds^2 = c^2 dt^2 - (d\vec{x})^2 = dt^2\left(1 - \beta^2(t)\right) \Rightarrow$

$$\Rightarrow ds = c\,dt\sqrt{1-\beta^2(t)} \Rightarrow \mathcal{S} = -Ac\int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}.$$

$\Rightarrow$ as $\mathcal{S} = \int_{t_1}^{t_2} dt \cdot L$, where L is the Lagrangian.