Last time | Four-vectors

Def. A^μ : $\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu}$ contravariant 4-vector

$\mu, \nu = 0, 1, 2, 3$

OR $A'^\mu = \Lambda^\mu{}_\nu A^\nu$

=> always sum over repeated indices (one upper, one lower)

Def. B_μ : $\boxed{B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu}$ covariant 4-vector

or $B_\mu = \Lambda_\mu{}^\nu B_\nu$

Def. $A_\mu B^\mu \equiv A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$

scalar product : it is Lorentz-invariant

Def. Metric tensor $g_{\mu\nu}$: $\underbrace{ds^2}_{\text{interval}} = g_{\mu\nu} dx^\mu dx^\nu$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ (Minkowski space)

Def. $g^{\mu\nu}$ by : $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$

$$\Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}$$

$A^\mu = g^{\mu\nu} A_\nu$ and $A_\mu = g_{\mu\nu} A^\nu$ ~ metric tensor raises & lowers indices

Examples: $x^\mu = (ct, \vec{x})$, $x_\mu = (ct, -\vec{x})$

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ $\Rightarrow$ $\partial_\mu \varphi$ is a covariant 4-vector

$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$ $\Rightarrow$ $\partial^\mu \varphi$ —"— contravariant —"—

$\varphi \sim$ a scalar

$\partial_\mu A^\mu$ = Lorentz invariant

$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is also Lorentz-invariant

$\Rightarrow$ wave eq'n operator is Lorentz-invariant!

4-velocity

Def. $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau}}$ $\Rightarrow$ $\boxed{u^\mu = \gamma\,(c, \vec{v})}$, $u_\mu u^\mu = c^2$.

But: time is not a scalar!

$$\frac{dx^\mu}{dt} \sim \frac{dx^\mu}{dx^0} \sim \text{not a Lorentz-vector.}$$

$\Rightarrow$ try proper time $d\tau = \frac{ds}{c} \Rightarrow u^\mu \equiv \frac{dx^\mu}{d\tau}$ 4-velocity

$$\text{as } d\tau = \frac{dt}{\gamma} \Rightarrow u^0 = \frac{c\,dt}{dt/\gamma} = c\gamma$$

$$\vec{u} = \frac{d\vec{x}}{dt}\cdot\gamma = \gamma\cdot\vec{v} \Rightarrow u^\mu = \gamma(c, \vec{v}).$$

Note $u_\mu u^\mu = c^2$.

Boost in terms of rapidity.

$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$\Rightarrow$ Define rapidity η

by $\beta \equiv \tanh\eta = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}$

Define light-cone coordinates $A^+ = \frac{A^0 + A^1}{\sqrt{2}}$

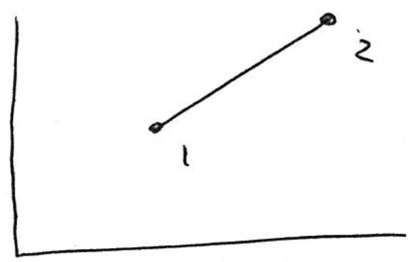
$$A^- = \frac{A^0 - A^1}{\sqrt{2}}$$

$\Rightarrow$ then

$$A_\mu A^\mu = 2A^+A^- - (A^2)^2 - (A^3)^2$$

Relativistic Mechanics.

Consider a free particle (moving along a straight line). We need to construct a Lorentz-invariant action for such a particle. It's characterized by a 4-vector x^μ $\Rightarrow$ the only Lorentz-invariant is the interval $\Rightarrow$



$\Rightarrow \int_1^2 ds$ (can't have $\int_1^2 (ds)^2$ ~ still infinitesimal)

$\Rightarrow$ write the action as $S = -A \cdot \int_1^2 ds$

As $ds^2 = c^2 dt^2 - (d\vec{x})^2 = dt^2 (1 - \beta^2(t)) \Rightarrow$

$\Rightarrow ds = c\,dt \sqrt{1-\beta^2(t)} \Rightarrow S = -Ac \int_{t_1}^{t_2} dt \sqrt{1-\beta^2(t)}$.

$\Rightarrow$ as $S = \int_{t_1}^{t_2} dt \cdot L$, where L is the <u>Lagrangian</u>,

$$\Rightarrow L = -Ac\sqrt{1-\beta^2(t)}\,.$$

Now, in classical non-relativistic mechanics we know that $L = K - V$

K ← kinetic energy, V ← potential energy

$\Rightarrow$ for a free NR particle $V=0 \Rightarrow L = K = \frac{1}{2}mv^2$

(We know that in non-relativistic (NR) limit: $K = \frac{1}{2}mv^2$.) $\Rightarrow$ as $\beta \to 0$ $\Rightarrow$

$$\Rightarrow L = \underbrace{-Ac}_{} + Ac\tfrac{1}{2}\beta^2 + \ldots$$

constant ~ drop, not important for dynamics

$$\Rightarrow A\cancel{c}\tfrac{\cancel{1}}{\cancel{2}}\frac{\cancel{v^2}}{c^2} = \tfrac{\cancel{1}}{\cancel{2}}m\cancel{v^2} \Rightarrow A = mc$$

$$\Rightarrow \boxed{S = -mc\int_1^2 ds = -mc^2\int_{t_1}^{t_2} dt\,\sqrt{1-\beta^2(t)}}$$

$$\boxed{L = -mc^2\sqrt{1-\beta^2(t)}}$$

action and Lagrangian for a free point particle

The particle's <u>Energy & Momentum.</u>

The free particle's degrees of freedom are coordinates $\vec{x}$ & t. <u>Momentum</u> is defined by: $p^i = \dfrac{\partial L}{\partial \dot{x}^i}$, where $i = 1, 2, 3$ and $\dot{x}^i = \dfrac{dx^i}{dt}$.

(know from classical mechanics).

$$\Rightarrow \quad p^i = \frac{\partial L}{\partial v^i} = -mc^2 \frac{-\cancel{2} v^i/c^2}{\cancel{2}\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m \vec{v}$$

<u>Energy</u> is defined by $E = \vec{p} \cdot \dot{\vec{x}} - L =$

$$= \vec{p} \cdot \vec{v} - L = \gamma m v^2 + mc^2 \sqrt{1-\frac{v^2}{c^2}} =$$

$$= \gamma \left[\cancel{mv^2} + mc^2 \left(1 - \cancel{\frac{v^2}{c^2}}\right)\right] = mc^2 \gamma$$

$$\Rightarrow \quad E = mc^2 \gamma = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$

(world's most famous formula)

Remember that $u^\mu = \gamma(c, \vec{v})$ – 4-velocity.

We now see that $\left(\frac{E}{c}, \vec{p}\right) = m\gamma(c, \vec{v}) \Rightarrow$

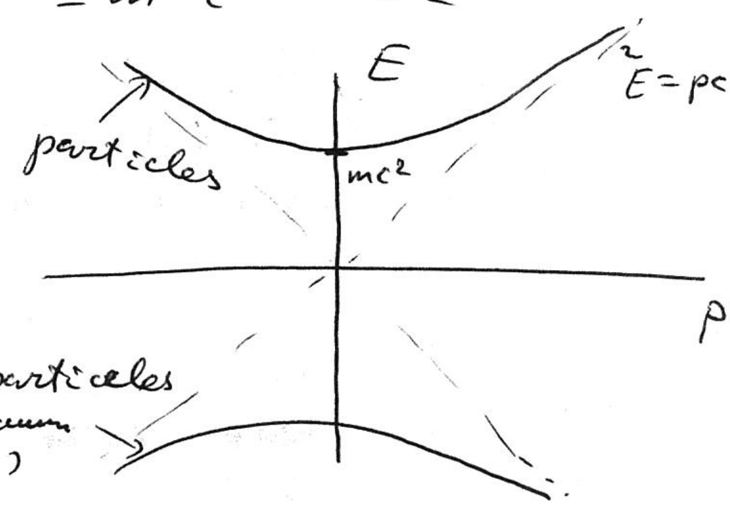
$\Rightarrow$ we have a new <u>4-momentum</u> four-vector:

$\boxed{p^\mu = m u^\mu}$, where $p^0 = \frac{E}{c}$, $p^i = (\vec{p})^i$

Note that $p_\mu p^\mu = m^2 u_\mu u^\mu = m^2 \gamma^2 (c^2 - v^2) =$

$= m^2 c^2 \Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$ or

$$\boxed{E^2 = \vec{p}^{\,2} c^2 + m^2 c^4}$$



4-vector p^μ transforms in the usual way:

$$\boxed{\begin{aligned} p^0 &= \gamma(p^{0\prime} + \beta p^{x\prime}) \\ p^x &= \gamma(p^{x\prime} + \beta p^{0\prime}) \\ p^y &= p^{y\prime}, \quad p^z = p^{z\prime} \end{aligned}}$$

(boost in x-direction)

Def.

<u>kinetic energy</u>

$T = E(v) - E(0) =$
$= mc^2[\gamma_u - 1]$

Newtonian mechanics: $\vec{F} = \frac{d\vec{p}}{dt}$ (force)

$\Rightarrow$ define force as $f^\mu = \frac{dp^\mu}{d\tau}$

$\Rightarrow \vec{f} = \underbrace{\frac{d\vec{p}}{dt}\gamma}_{``\vec{F}\cdot\gamma}$ $\Rightarrow$ in NR limit gives Newtonian result.

$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt}$; Note that $f^\mu u_\mu = 0$

$$\left(u_\mu f^\mu = u_\mu \frac{dp^\mu}{d\tau} = u_\mu m \frac{du^\mu}{d\tau} = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = \right.$$

$$\left. = \frac{1}{2} m \frac{dc^2}{d\tau} = 0 \right) \Rightarrow f^0 \cdot u^0 = \vec{f} \cdot \gamma \vec{v} \Rightarrow$$

$$\Rightarrow f^0 c = \vec{f}\cdot\vec{v} \Rightarrow f^0 = \frac{\vec{f}\cdot\vec{v}}{c} \Rightarrow \gamma \frac{dp^0}{dt} = f^0 = \frac{\vec{f}\cdot\vec{v}}{c}$$

$$\Rightarrow \gamma \frac{dE}{dt} = \vec{f}\cdot\vec{v} = \gamma\, \vec{F}\cdot\vec{v} \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

($\vec{F}$ is Newtonian NR force).

$\Rightarrow$ 4-momentum is conserved in particle interactions.

$$\boxed{\sum p^\mu_{initial} = \sum p^\mu_{final}}$$