Last time: Relativistic Mechanics (cont'd)

free point particle: $\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}}}$ Lagrangian

The action is $\boxed{S = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1-\frac{v^2}{c^2}}}$.

Momentum: $p^i \equiv \frac{\partial L}{\partial \dot{x}^i}$ ($\frac{\partial L}{\partial \dot{x}^i}$ is certainly a 3-vector; whether it is a 3-vector part of a covariant or a contravariant 4-vector remains undetermined at this point $\Rightarrow$ hence we just denote the 3-vector p^i. In classical mechanics $\vec{p}$ is conjugate to $\vec{q}\ (\equiv \vec{x})$.)

$\Rightarrow \quad \boxed{\vec{p} = \gamma m \vec{v}}$

Energy $\quad E = H = \vec{p}\cdot\dot{\vec{x}} - L \Rightarrow \boxed{E = \gamma mc^2}$

$\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right) = mu^\mu} \Rightarrow$ a 4-vector of momentum (4-momentum)

$p_\mu p^\mu = m^2c^2 \Rightarrow \boxed{E^2 = p^2c^2 + m^2c^4}$.

Remember that $u^\mu = \gamma(c, \vec{v})$ ~ 4-velocity.

We now see that $\left(\frac{E}{c}, \vec{p}\right) = m\gamma(c, \vec{v}) \Rightarrow$

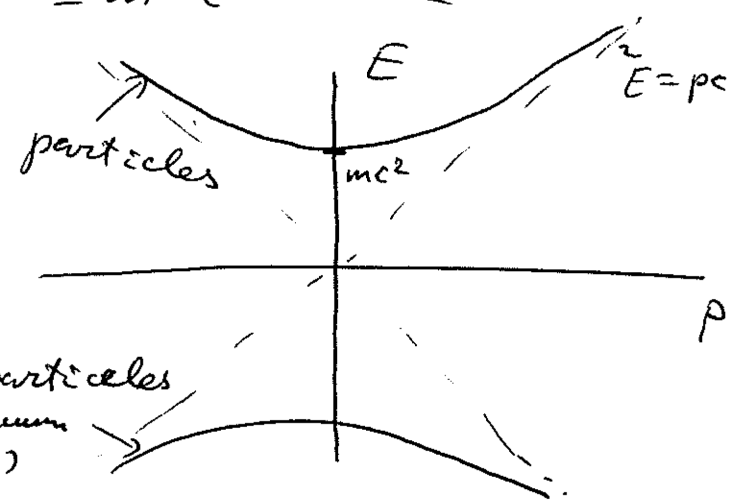
$\Rightarrow$ we have a new 4-momentum four-vector:

$\boxed{p^\mu = m u^\mu}$, where $p^0 = \frac{E}{c}$, $p^i = (\vec{p})^i$

Note that $p_\mu p^\mu = m^2 u_\mu u^\mu = m^2 \gamma^2 (c^2 - v^2) =$

$= m^2 c^2 \Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$ or

$$\boxed{E^2 = \vec{p}^{\,2} c^2 + m^2 c^4}$$



4-vector p^μ transforms in the usual way:

$$\boxed{\begin{aligned} p^0 &= \gamma(p^{0\prime} + \beta p^{x\prime}) \\ p^x &= \gamma(p^{x\prime} + \beta p^{0\prime}) \\ p^y &= p^{y\prime}, \quad p^z = p^{z\prime} \end{aligned}}$$

(boost in x-direction)

Def. kinetic energy

$T = E(v) - E(0) =$
$= mc^2 [\gamma_u - 1]$

Newtonian mechanics: $\vec{F} = \frac{d\vec{p}}{dt}$ (force)

$\Rightarrow$ define force as $f^\mu = \frac{dp^\mu}{d\tau}$

$\Rightarrow \vec{f} = \underbrace{\frac{d\vec{p}}{dt}\gamma}_{\text{"}\vec{F}\cdot\gamma}$ $\Rightarrow$ in NR limit gives Newtonian result.

$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt}$; Note that $f^\mu u_\mu = 0$

$$\left(u_\mu f^\mu = u_\mu \frac{dp^\mu}{d\tau} = u_\mu m \frac{du^\mu}{d\tau} = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = \right.$$

$$\left. = \frac{1}{2} m \frac{dc^2}{d\tau} = 0 \right) \Rightarrow f^0 \cdot u^0 = \vec{f} \cdot \gamma \vec{v} \Rightarrow$$

$$\Rightarrow f^0 c = \vec{f} \cdot \vec{v} \Rightarrow f^0 = \frac{\vec{f}\cdot\vec{v}}{c} \Rightarrow \gamma \frac{dp^0}{dt} = f^0 = \frac{\vec{f}\cdot\vec{v}}{c}$$

$$\Rightarrow \gamma \frac{dE}{dt} = \vec{f}\cdot\vec{v} = \gamma \vec{F}\cdot\vec{v} \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

($\vec{F}$ is Newtonian NR force).

$\Rightarrow$ 4-momentum is conserved in particle interactions.

$$\boxed{\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}}$$

~~is $s \geq s_{min} = c^2 (m_1 + \ldots + m_n)^2$~~

~~Threshold energy~~

Covariance of Electrodynamics

Start with Maxwell equations in vacuum: (microscopic Maxwell eqns)

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi\rho & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} & \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{cases}$$

(in Gaussian units now, in vacuum $\vec{B} = \vec{H}$, $\vec{E} = \vec{D}$).

$\left(\vec{D} = \vec{E} + 4\pi \vec{P}, \quad \vec{H} = \vec{B} - 4\pi \vec{M} \text{ in general} \right)$

Define scalar and vector potentials by

$$\vec{E} = -\vec{\nabla} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Pick Lorenz gauge condition: $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

$\Rightarrow$ Gauss's law gives: $-\nabla^2 \Phi - \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = 4\pi\rho$

$$\Rightarrow \quad \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho.$$

Ampere's law leads to: $\cancel{\vec{\nabla}(\vec{\nabla} \cdot \vec{A})} - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J} -$

$$\cancel{-\frac{1}{c} \vec{\nabla} \frac{\partial \Phi}{\partial t}} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \Rightarrow \quad \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}.$$

Before we continue, let's establish transformation properties of $\vec{J}, \rho$: we know that $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ (continuity) $\Rightarrow$ defining an object $J^\mu \equiv (c\rho, \vec{J})$

we write the condition as $\boxed{\partial_\mu J^\mu = 0}$

$\partial_\mu \sim$ covariant 4-vector

If continuity holds in all frames $\Rightarrow \partial_\mu J^\mu$ is Lorentz-scalar $\Rightarrow J^\mu$ is a contravariant 4-vector!

<u>Charge conservation</u>: the charge in a volume element d^3x is the same in any frame:

in some frame K have $\rho(\vec{x}, t)\, d^3x$

in another frame K' have $\rho'(\vec{x}', t')\, d^3x'$

$\Rightarrow \rho\, d^3x = \rho'\, d^3x'$

Is this correct?

$\Rightarrow$ Note that $d^4x = c\,dt\, d^3x = dx^0 d^3x$ is a Lorentz-scalar: take a general Lorentz-transform, $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow d^4x' = d^4x \cdot \underset{\text{Jacobian}}{\det \Lambda}$

$$\Rightarrow \quad \det \Lambda = \det \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2 - \beta^2\gamma^2 = 1$$

$\Rightarrow d^4x' = d^4x \Rightarrow dx'^0 d^3x' = dx^0 d^3x \Rightarrow$

for $\rho d^3x = \rho' d^3x' \Rightarrow$ need ρ to transform like x^0

$\Rightarrow$ zero-component of a 4-vector.

$\Rightarrow$ Current density $\vec{J} = \rho \vec{v} = \rho \frac{d\vec{x}}{dt} \Rightarrow$ if

$\rho \sim x^0$, $t \sim x^0 \Rightarrow \vec{J}$ is a 3-vector.

Back to Maxwell equations in Lorenz gauge:

as $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu \Rightarrow$ it is a Lorentz-scalar and Maxwell equations become:

$$\begin{cases} \partial_\mu \partial^\mu \Phi = 4\pi\rho & \Rightarrow \Phi \sim \rho \sim x^0 \\ \partial_\mu \partial^\mu \vec{A} = \frac{4\pi}{c}\vec{J} & \Rightarrow \vec{A} \sim \vec{J} \sim \vec{x} \end{cases}$$

with Lorentz gauge condition $\left(\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0\right)$.

$\Rightarrow$ we can define a 4-vector potential: $\boxed{A^\mu = (\Phi, \vec{A})}$

Now Lorenz gauge condition is $\partial_\mu A^\mu = 0$

– manifestly covariant, sometimes called covariant gauge. (cf. $\vec{\nabla}\cdot\vec{A} = 0$ Coulomb gauge)

Maxwell equations become $\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$

or, defining $\Box \equiv \partial_\mu \partial^\mu$, $\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$.

Now let's express $\vec{E}$ and $\vec{B}$ in this covariant notation: $\vec{E} = -\frac{\partial \vec{A}}{\partial x^0} - \vec{\nabla}\Phi \Rightarrow E^i = -\frac{\partial A^i}{\partial x^0} - \frac{\partial A^0}{\partial x^i} =$

$\Rightarrow E^i = -(\partial^0 A^i - \partial^i A^0)$

$= \varepsilon^{ijk}\partial_j A^k$

$B^i = -\varepsilon^{ijk}\partial^j A^k \Rightarrow B^1 = -(\partial^2 A^3 - \partial^3 A^2)$. $\left(B_z = \partial_x A_y - \partial_y A_x = \partial_1 A^2 - \partial_2 A^1 \Rightarrow B^3 = -(\partial^1 A^2 - \partial^2 A^1)\right)$

$\Rightarrow$ Define field-strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \quad \Rightarrow \quad E^i = -F^{0i}, \quad B^i = -\frac{1}{2}\varepsilon^{ijk} F^{jk}$$

anti-symmetric ($F^{\mu\nu} = -F^{\nu\mu}$), 2nd rank tensor

$$\Rightarrow F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{0i} = -E^i, \quad F^{ij} = -\varepsilon^{ijk} B^k$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\Rightarrow$ Gauss's law & Ampere's law can be summarized by $\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$

(in Lorenz gauge $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{=0 \text{ as } \partial_\mu A^\mu = 0} =$

$= \Box A^\nu = \frac{4\pi}{c} J^\nu$ ~ ok)

To write Faraday's law and $\vec{\nabla}\cdot\vec{B} = 0$ in a similar fashion, define a dual tensor

$$\boxed{\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

where $\varepsilon^{0123} = 1$, $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma}$ ~ changes sign under permutations and $\varepsilon^{\mu\alpha\rho\alpha} = \varepsilon^{\alpha\alpha\rho\sigma} = \ldots = 0$ (as 2 indices are the same)

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

duality transform
$\vec{E} \to \vec{B}$
$\vec{B} \to -\vec{E}$.

$\Rightarrow$ the Faraday's law & $\vec{\nabla}\cdot\vec{B} = 0$ are written by $\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$

(can one prove this?)

Thus the Maxwell equations now become

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

(the second one is really needed to define $\vec{E}, \vec{B}$ in terms of A_μ $\Rightarrow$ only the 1st one is called Maxwell eqn's usually).

Transformation of $\vec{E}$ & $\vec{B}$ under Boosts.

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} F^{\mu\nu} \Lambda_\nu{}^{\nu'}$$

as $\Lambda^{\nu'}{}_{\nu} = \Lambda_\nu{}^{\nu'}$ or $\Lambda = \Lambda^T$ (for boosts only)

$$\Rightarrow F^{\mu'\nu'} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (\beta^2\gamma^2 - \gamma^2)E_1 & -\gamma E_2 + \beta\gamma B_3 & -\gamma E_3 - \beta\gamma B \\ (\gamma^2 - \beta^2\gamma^2)E_1 & 0 & \beta\gamma E_2 - \gamma B_3 & \beta\gamma E_3 + \gamma B \\ \gamma E_2 - \beta\gamma B_3 & -\beta\gamma E_2 + \gamma B_3 & 0 & -B_1 \\ \gamma E_3 + \beta\gamma B_2 & -\beta\gamma E_3 - \gamma B_2 & B_1 & 0 \end{pmatrix}$$

$\Rightarrow$