Last time

$$f^\mu = \frac{dp^\mu}{d\tau}$$ ~ 4-vector of force

$f^0 = \frac{\vec{f}\cdot\vec{v}}{c}$, $\vec{f} = \gamma\vec{F}$, such that

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v}, \quad \frac{d\vec{p}}{dt} = \vec{F}$$ ~ usual NR force

Covariance of Electrodynamics

Def. Current 4-vector: $J^\mu = (c\rho, \vec{J})$

$$\partial_\mu J^\mu = 0$$ current conservation (continuity)

Def. 4-vector potential $A^\mu = (\Phi, \vec{A})$

Def. Field strength tensor $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

with $F^{0i} = -E^i$, $F^{ij} = -\varepsilon^{ijk} B^k$

Def. Dual field strength tensor

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$\Rightarrow$ Maxwell equations are

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

(Gauss's & Ampere's law)

(Faraday's law & no monopoles)

in Lorenz gauge $\partial_\mu A^\mu = 0$

$\Rightarrow$ Maxwell eqn's are

$$\Box A^\mu = \frac{4\pi}{c} J^\nu$$

where $\Box \equiv \partial_\mu \partial^\mu$.

$\Rightarrow$ Gauss's law & Ampere's law can be summarized by $\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$

(in Lorenz gauge $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{=0 \text{ as } \partial_\mu A^\mu = 0} =$

$= \Box A^\nu = \frac{4\pi}{c} J^\nu$ ~ ok)

To write Faraday's law and $\vec{\nabla}\cdot\vec{B} = 0$ in a similar fashion, <u>define</u> a dual tensor

$$\boxed{\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

where $\varepsilon^{0123} = 1$, $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma}$ ~ changes sign under permutations and $\varepsilon^{\mu\alpha\rho\alpha} = \varepsilon^{\alpha\alpha\rho\sigma} = \ldots = 0$ (no 2 indices are the same)

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

duality transform
$\vec{E} \to \vec{B}$
$\vec{B} \to -\vec{E}$.

$\Rightarrow$ the Faraday's law & $\vec{\nabla}\cdot\vec{B} = 0$ are written by $\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$

(can one prove this?)

Thus the Maxwell equations now become

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

(the second one is really needed to define $\vec{E}, \vec{B}$ in terms of A_μ $\Rightarrow$ only the 1st one is called Maxwell equ's usually).

Transformation of $\vec{E}$ & $\vec{B}$ under Boosts.

$$F'^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} F^{\mu\nu} \Lambda_\nu{}^{\nu'}$$

as $\Lambda^{\nu'}{}_{\nu} = \Lambda_\nu{}^{\nu'}$ or $\Lambda = \Lambda^T$ (for boosts only)

$$\Rightarrow F'^{\mu'\nu'} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (\beta^2\gamma^2 - \gamma^2)E_x & -\gamma E_y + \beta\gamma B_z & -\gamma E_z - \beta\gamma \ldots \\ (\gamma^2 - \beta^2\gamma^2)E_x & 0 & \beta\gamma E_y - \gamma B_z & \beta\gamma E_z + \gamma B \ldots \\ \gamma E_y - \beta\gamma B_z & -\beta\gamma E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z + \beta\gamma B_y & -\beta\gamma E_z - \gamma B_y & B_x & 0 \end{pmatrix}$$

$$\Rightarrow \text{as} \quad F'^{\mu'\nu'} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x & 0 & -B_z' & B_y' \\ E_y & B_z' & 0 & -B_x' \\ E_z & -B_y' & B_x' & 0 \end{pmatrix} \Rightarrow$$

$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

if $v \ll c \Rightarrow$ get $\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$

$\vec{B}' = \vec{B} - \frac{1}{c}\vec{v} \times \vec{E}$. (Galilean transformations for $\vec{E}$ & $\vec{B}$)

Lorentz - invariants:

$F^{\mu\nu} F_{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$ ~ by construction this is Lorentz-inv.

$F^{\mu\nu} \tilde{F}_{\mu\nu} = -4\, \vec{B} \cdot \vec{E}$ ~ also Lorentz-inv.

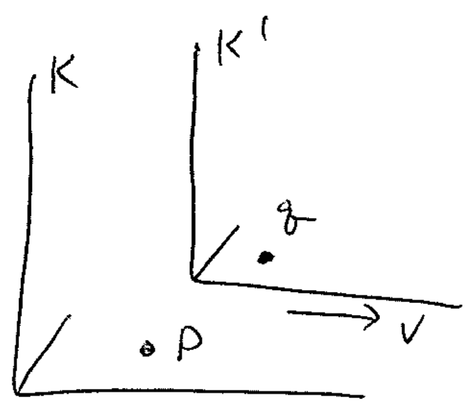
<u>Example</u>: plane waves, $\vec{E} = -\frac{c}{\omega}\vec{k} \times \vec{B} \Rightarrow$

$\Rightarrow \vec{E} \cdot \vec{B} = 0$, $|\vec{E}| = |\vec{B}| \Rightarrow E^2 - B^2 = 0$

~ true in all frames!

<u>Example</u>: moving point charge: in it's rest frame the field is given by Coulomb's law: $\vec{E}' = \frac{q}{r'^3}\vec{r}'$

(note Gaussian units)



$\Rightarrow$ observer at P has coordinates (x_1', x_2', x_3')

$\Rightarrow$ $\vec{E}'(P) = q \frac{\vec{x}'}{\vec{x}'^3}$ (assume that charge q is at the origin in frame K')

$\vec{B}' = 0$

$\Rightarrow$ boost $\rightarrow$ $E_x = E_x' = q \frac{x'}{(x'^2 + y'^2 + z'^2)^{3/2}} =$

$$= q \frac{\gamma(x - vt)}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_y = \gamma(E_y' + \beta \underbrace{B_z'}_{0}) = q \frac{\gamma y}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_z = \gamma(E_z' - \beta \underbrace{B_y'}_{=0}) = q \frac{\gamma z}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$B_x = B_x' = 0; \quad B_y = \gamma(\overset{=0}{B_z'} - \beta E_z') =$$

$$= -\gamma\beta \frac{q z}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$B_z = \gamma(\underbrace{B_z'}_{=0} + \beta E_y') = \gamma\beta \frac{q y}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

In the non-relativistic case: $\beta \ll 1$, $\gamma \simeq 1$

$$\Rightarrow B_y \simeq -\frac{v}{c} \frac{q z}{[(x-vt)^2 + y^2 + z^2]^{3/2}}; \quad B_z \simeq \frac{v}{c} \frac{q y}{[(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$\Rightarrow \vec{B} = \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3} \Rightarrow \text{Biot-Savart Law!}$$

(as expected!)

$$\lim_{\gamma\to\infty} \frac{\gamma}{\left(\gamma^2 \underbrace{(x-ct)}_{\xi}{}^2 + x_\perp^2\right)^{3/2}} = \begin{cases} 0, & x \neq ct \\ \infty, & x = ct \end{cases} = \frac{2}{x_\perp^2 \gamma}\,\delta(x-ct)$$

$$x_\perp^2 = y^2 + z^2$$

$$\int_{-\infty}^{\infty} d\xi \, \frac{\gamma}{\left[\gamma^2 \xi^2 + x_\perp^2\right]^{3/2}} = \left| \tilde{\xi} = \gamma \xi \right| = \int_{-\infty}^{\infty} \frac{d\tilde{\xi}}{\left[\tilde{\xi}^2 + x_\perp^2\right]^{3/2}} = \frac{2}{x_\perp^2}$$

$$\Rightarrow \boxed{\begin{aligned} E_y &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_z &= q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_x &= 0 \qquad B_x = 0 \\ B_y &= -q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ B_z &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \end{aligned}}$$

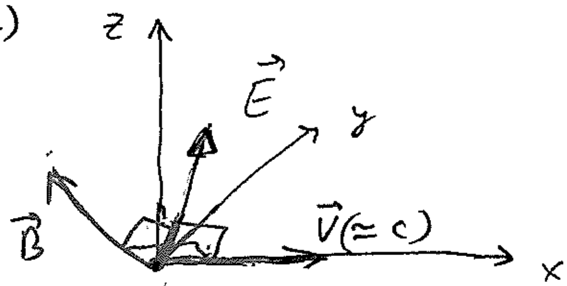
as $\gamma \to \infty$

$\beta \to 1$

$$\underline{E} = 2q\,\frac{\underline{X}}{\underline{X}^2}\,\delta(x-ct), \quad \underline{X} \equiv (y, z)$$

$$\vec{B} = \hat{x} \times \vec{E} \left(= \frac{c}{\omega}\,\vec{k} \times \vec{E} \right)$$

for $\vec{k} = \frac{\omega}{c}\hat{x}$.



~ looks like a plane wave frozen around the particle. ⇒ equivalent photon approximation (same story for quarks & gluons in a proton)