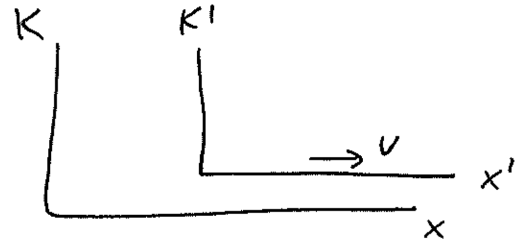
Last time | Transformation of $\vec{E}$ & $\vec{B}$ under boosts (cont'd)

used the fact that $F^{\mu\nu}$ is a Lorentz tensor to infer the following:



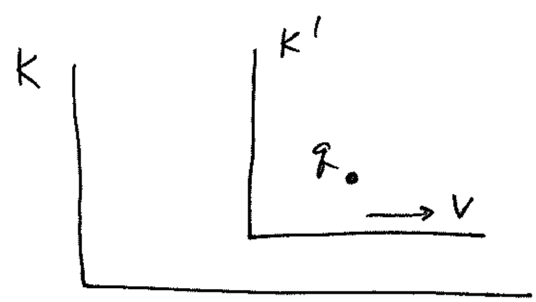
$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

Lorentz invariants: $F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\, \vec{B} \cdot \vec{E}$$

Example | Point charge q moving with velocity v:

Using $\vec{E}$ & $\vec{B}$ field transformations from above we found:



$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{q\gamma}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \begin{pmatrix} x - vt \\ y \\ z \end{pmatrix}$$

along with

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{q\gamma\beta}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}.$$

$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

if $v \ll c \Rightarrow$ get
$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$$
$$\vec{B}' = \vec{B} - \frac{1}{c}\vec{v} \times \vec{E}.$$
(Galilean transformations for $\vec{E}$ & $\vec{B}$)

Lorentz - invariants:

$F^{\mu\nu} F_{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$ ~ by construction this is Lorentz-inv.

$F^{\mu\nu} \tilde{F}_{\mu\nu} = -4\,\vec{B}\cdot\vec{E}$ ~ also Lorentz-inv.

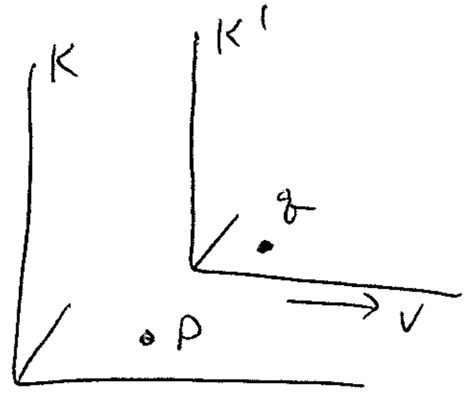
<u>Example</u>: plane waves, $\vec{E} = -\frac{c}{\omega}\vec{k} \times \vec{B} \Rightarrow$

$\Rightarrow \vec{E}\cdot\vec{B} = 0$, $|\vec{E}| = |\vec{B}| \Rightarrow E^2 - B^2 = 0$

~ true in all frames!

<u>Example</u>: moving point charge: in it's rest frame the field is given by Coulomb's law: $\vec{E}' = \frac{q}{r'^3}\vec{r}'$

(note Gaussian units)



$\Rightarrow$ observer at P has coordinates (x_1', x_2', x_3')

$\Rightarrow$ $\vec{E}'(P) = q \dfrac{\vec{x}'}{\vec{x}'^3}$ (assume that charge q is at the origin in frame K')

$\vec{B}' = 0$

$\Rightarrow$ boost $\rightarrow$ $E_x = E_x' = q \dfrac{x'}{(x'^2 + y'^2 + z'^2)^{3/2}} =$

$$= q \frac{\gamma(x - vt)}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_y = \gamma(E_y' + \beta \underbrace{B_z'}_{0}) = q \frac{\gamma y}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_z = \gamma(E_z' - \beta \underbrace{B_y'}_{=0}) = q \frac{\gamma z}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$B_x = B_x' = 0$; $B_y = \gamma(\overset{=0}{B_z'} - \beta E_z') =$

$$= -\gamma\beta \frac{q z}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$B_z = \gamma(\underbrace{B_z'}_{=0} + \beta E_y') = \gamma\beta \frac{q y}{[\gamma^2(x-vt)^2 + y^2 + z^2]^{3/2}}$$

In the non-relativistic case: $\beta \ll 1$, $\gamma \simeq 1$

$$\Rightarrow B_y \simeq -\frac{v}{c} \frac{q z}{[(x-vt)^2 + y^2 + z^2]^{3/2}}; \quad B_z \simeq \frac{v}{c} \frac{q y}{[(x-vt)^2 + y^2 + z^2]^{3/2}}$$

$$\Rightarrow \vec{B} = \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3} \quad \Rightarrow \text{Biot-Savart Law!}$$

(as expected!)

$$\lim_{\gamma\to\infty} \frac{\gamma}{\left(\gamma^2\underbrace{(x-ct)}_{\xi}{}^2 + x_\perp^2\right)^{3/2}} = \begin{cases} 0, & x \neq ct \\ \infty, & x = ct \end{cases} = \frac{2}{x_\perp^2\,\gamma}\,\delta(x-ct)$$

$$x_\perp^2 = y^2 + z^2$$

$$\int_{-\infty}^{\infty} d\xi \, \frac{\gamma}{\left[\gamma^2\xi^2 + x_\perp^2\right]^{3/2}} = \left|\, \tilde{\xi} = \gamma\xi \,\right. = \int_{-\infty}^{\infty} \frac{d\tilde{\xi}}{\left[\tilde{\xi}^2 + x_\perp^2\right]^{3/2}} = \frac{2}{x_\perp^2}$$

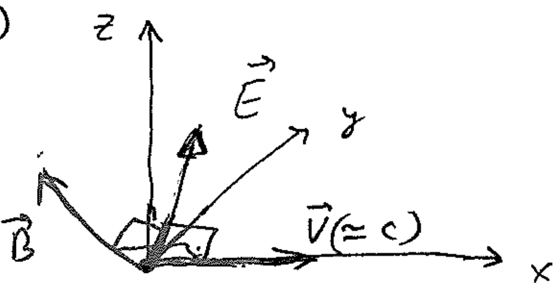
$$\Rightarrow \boxed{\begin{aligned} E_y &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_z &= q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_x &= 0 \qquad B_x = 0 \\ B_y &= -q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ B_z &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \end{aligned}}$$

as $\gamma \to \infty$

$\beta \to 1$

$$\underline{E} = 2q\,\frac{\underline{X}}{\underline{X}^2}\,\delta(x-ct), \quad \underline{X} \equiv (y, z)$$

$$\vec{B} = \hat{x} \times \vec{E} \quad \left(= \frac{c}{\omega}\,\vec{k} \times \vec{E}\right) \quad \text{for } \vec{k} = \frac{\omega}{c}\hat{x}.$$



→ looks like a plane wave frozen around the particle. ⇒ equivalent photon approximation (same story for quarks & gluons in a proton)

$$\Rightarrow \vec{B} = \frac{q}{c}\frac{\vec{V}\times\vec{r}}{r^3} \sim \text{Biot-Savart law!}$$

In the UR limit, $\beta \to 1$, $\gamma \to \infty$ $\Rightarrow$

$$\Rightarrow B_y \approx -\frac{q z}{\gamma^2 (x-vt)^3} = -E_z \; ; \quad B_z \approx \frac{q y}{\gamma^2 (x-vt)^3} = E_y \; ;$$

$$\sim -qz\,\delta(x-\tfrac{v}{c}t) \qquad \sim qy\,\delta(x-ct)$$

$$E_x \approx \frac{q}{\gamma^2 (x-vt)^2}\,\mathrm{Sign}(x-vt) \to 0$$

see attached

Relativistic Particles in Electromagnetic Fields.

Start with Lorentz force:

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\right) = \vec{F}$$

$\Rightarrow$ want to transform this into an equation for 3-components of 4-vector p^μ $\Rightarrow$

$$\Rightarrow p^\mu = m u^\mu = m\gamma(c, \vec{V}) \Rightarrow$$

$$F^{ij} = -\varepsilon^{ijk}B^k$$

$$\frac{d\vec{p}^i}{d\tau} = \gamma\frac{d\vec{p}^i}{dt} = \frac{q}{c}\left[u^0\vec{E} + \vec{u}\times\vec{B}\right]^i = \frac{q}{c}F^{i\mu}u_\mu$$

$$E^i = F^{i0} \Rightarrow u^0F^{i0} + \varepsilon^{ijk}u^jB^k = u^0F^{i0} - F^{ij}u^j = u_0F^{i0} + u_jF^{ij} = u_\mu F^{i\mu}.$$

On the other hand $\dfrac{dE}{dt} = \vec{F}\cdot\vec{V}$ $\Rightarrow$

$$\Rightarrow \frac{dp^0}{d\tau} = \gamma \frac{1}{c} \frac{dE}{dt} = \frac{\gamma}{c} q \vec{v} \cdot \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) =$$

$$= \frac{q}{c} \gamma \vec{v} \cdot \vec{E} \quad \Rightarrow \text{as } E^i = -F^{0i} \Rightarrow$$

$$\Rightarrow \gamma \vec{v} \cdot \vec{E} = -\gamma v_i E^i = \gamma v_i F^{0i} = u_i F^{0i} =$$

$$= u_\mu F^{0\mu} \text{ as } F^{00} = 0 \Rightarrow \frac{dp^0}{d\tau} = \frac{q}{c} u_\mu F^{0\mu}$$

Also, $\frac{dp^i}{d\tau} = \frac{q}{c} u_\mu F^{i\mu}$ (from above)

$$\Rightarrow \boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c} u_\nu F^{\mu\nu}}$$

Lorentz-covariant formulation of Lorentz force.

$\Rightarrow$ What does this mean for the Lagrangian?

Remember, $L_{free} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$.

If an action is given by $S = \int_{t_1}^{t_2} dt\, L[q_i(t), \dot{q}_i(t), t]$

$\Rightarrow$ least action principle gives

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.}$$

Apply this to L_{free}: $q_i(t) = x^i(t)$,

$\dot{q}_i(t) = \frac{d}{dt} x^i(t) = \dot{x}^i(t) = v^i(t) \Rightarrow L_{free} = L(\dot{x}^i)$

$\Rightarrow$ get $\frac{d}{dt}(m\gamma\vec{v}) = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0. \Rightarrow m\gamma\vec{v} = \text{const.}$ (momentum conservation)

Let's try to construct the interaction Lagrangian between point charges & E&M fields.

In the NR limit $L = T - V \Rightarrow$ the potential energy of a point charge in electric field is $V = e\Phi \Rightarrow L \simeq -e\Phi$

$\Rightarrow$ however, $\Phi = A^0$ and ~~area~~ we need to have a covariant expression $\Rightarrow \Phi \to A^\mu \Rightarrow$

$\Rightarrow$ need to multiply by a 4-vector $\Rightarrow$

$\Rightarrow$ only have u^μ (can't have x^μ - would have "prefferred" coordinates $\Rightarrow$ not physically meaningfull) $\Rightarrow$ get $L \overset{?}{\simeq} -\frac{e}{c} u_\mu A^\mu$.

Still, the action $S = \int dt \cdot L$ is Lorentz-scalar

$\Rightarrow S = \int d\tau \cdot \gamma \cdot L$ is a scalar $\Rightarrow \gamma L$ is a scalar $\Rightarrow$

$$\Rightarrow \quad L_{int} = -\frac{e}{c\gamma} u_\mu A^\mu$$

$$\Rightarrow \quad S_{int} = -\frac{e}{c}\int_{t_1}^{t_2} dt \, \frac{1}{\gamma} u_\mu A^\mu = -\frac{e}{c}\int_1^2 dx_\mu A^\mu$$

The total Lagrangian is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c\gamma} u_\mu A^\mu \Rightarrow$$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{m v^i}{\sqrt{1-\frac{v^2}{c^2}}}\right) + e\frac{\partial \Phi}{\partial x^i} -$$

$$- \frac{e}{c}\vec{v}\cdot\frac{\partial \vec{A}}{\partial x^i} + \frac{e}{c}\frac{dA^i}{dt} = 0 \Rightarrow \text{as } \frac{dA^i}{dt} = \frac{\partial A^i}{\partial t} + \frac{\partial x_j}{\partial t}\frac{\partial A^i}{\partial x_j}$$

$$\frac{dp^i}{dt} = -\frac{e}{c}\left[c\frac{\partial A^0}{\partial x^i} - v^j\frac{\partial A^j}{\partial x^i} + c\frac{\partial A^i}{\partial x^0} + \right.$$

$$\left. + v_j\frac{\partial A^i}{\partial x_j}\right] = -\frac{e}{c}\left[c\underbrace{\left(-\frac{\partial A^0}{\partial x_i} + \frac{\partial A^i}{\partial x^0}\right)}_{F^{0i}} - v_j\underbrace{\left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right)}_{F^{ij}}\right]$$

$$= \frac{e}{\gamma c} u_\mu F^{i\mu} \Rightarrow \frac{dp^i}{d\tau} = \frac{e}{c} u_\mu F^{i\mu}, \text{ as desired!}$$

$$\frac{dp^0}{d\tau}: \quad u_\mu \frac{dp^\mu}{d\tau} = 0 \quad \Rightarrow \quad \gamma c \frac{dp^0}{d\tau} - \gamma\, \vec{v}\cdot\frac{d\vec{p}}{d\tau} = 0$$

$$\Rightarrow \frac{dp^0}{d\tau} = \frac{\vec{v}}{c}\,\frac{d\vec{p}}{d\tau} = \frac{q}{c}\cdot\frac{v^i}{c}\, u_\mu F^{i\mu} = \frac{q}{c}\,\frac{v^i}{c}\cdot$$

$$\cdot\left[\gamma c F^{i0} - \underbrace{\gamma v^j F^{ij}}_{\to 0}\right] = \frac{q}{c}\,\gamma v^i F^{i0} =$$

$$= -\frac{q}{c}\,\gamma v^i F^{0i} = \frac{q}{c}\, u_\mu F^{0\mu}.$$

We know L $\Rightarrow$ can find canonical momentum

$\vec{P}$ by $P_i = \dfrac{\partial L}{\partial v_i} = \underbrace{\gamma m v_i}_{\text{momentum of a free particle}} + \dfrac{e}{c} A_i = \left(\vec{p} + \dfrac{e}{c}\vec{A}\right)_i$

$$\Rightarrow \boxed{\vec{P} = \vec{p} + \frac{e}{c}\vec{A}} \quad \Big(\text{in QM } \underset{\text{quantization}}{\vec{P} \to -i\hbar\vec{\nabla}}\Big)$$

Now we can find the Hamiltonian of the system:

$H = \vec{P}\cdot\vec{v} - L \Big|_{\substack{\text{with } \vec{v} \text{ expressed}\\ \text{in terms of } \vec{P}}} \Rightarrow$

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}$$

$$\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 = \frac{m^2 v^2}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow v^2\left(m^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 \frac{1}{c^2}\right) = \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2$$

$$\Rightarrow \vec{v} = \frac{c\left(\vec{P} - \frac{e}{c}\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}.$$

$$\Rightarrow H = \frac{\vec{P}\cdot\left(c\vec{P} - e\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} + e\Phi - \frac{e}{c}\frac{\vec{A}\cdot\left(c\vec{P} - e\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}$$

$$+ mc^2\sqrt{1 - \frac{\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = \frac{c\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} + e\Phi +$$

$$+ \frac{m^2c^3}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\Phi$$

$$\Rightarrow \boxed{H = \sqrt{m^2c^4 + (c\vec{P} - e\vec{A})^2} + e\Phi}$$ total energy of the particle

$$\text{"}\sqrt{m^2c^4 + p^2c^2}$$

Motion of a point charge in external $\vec{E}, \vec{B}$ fields:

We have Lorentz force $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$

and energy change $\frac{dE}{dt} = q\,\vec{v}\cdot\vec{E}$.

A. Constant Uniform Electric Field.

$$\frac{d\vec{p}}{dt} = q\vec{E} \Rightarrow \vec{p} = q\vec{E}t + \text{const} \Rightarrow \text{if}$$

the particle starts from rest $\Rightarrow \vec{p}|_{t=0} = 0$

$$\Rightarrow \vec{p} = q\vec{E}t \Rightarrow \text{is } \vec{E} = E\hat{x} \Rightarrow$$

$$\Rightarrow p_x = qEt, \quad p_y = p_z = 0$$

$$\Rightarrow \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = qEt \Rightarrow m^2\left(\frac{dx}{dt}\right)^2 = q^2E^2t^2\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t^2}}$$

$$\Rightarrow x(t) = \int_0^t dt' \frac{qEt'}{\sqrt{m^2 + \frac{q^2}{c^2}E^2 t'^2}} = \frac{\cancel{q}\cancel{E}}{\cancel{m}} \frac{c^2 m^{\cancel{2}}}{q^{\cancel{2}} E^{\cancel{2}}}\left(\sqrt{1 + \frac{q^2E^2}{m^2c^2}t^2} - 1\right)$$

assume $X(0)=0$

$$\Rightarrow \boxed{x(t) = \frac{mc^2}{qE}\left(\sqrt{1 + \frac{q^2E^2}{m^2c^2}t^2} - 1\right)}$$

moves with speed of light!

$\Rightarrow$ as $t \to \infty \Rightarrow x(t) \simeq c\ t.$ ~ linear in t!

$\Rightarrow$ if c is large $\Rightarrow$ expand in powers of $\frac{1}{c}$ $\Rightarrow$

$$\Rightarrow x(t) \simeq \frac{1}{2}\frac{qE}{m}t^2 = \frac{1}{2}at^2$$ ~ well-known classical NR result!

B. Constant Uniform Magnetic Field.

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}, \quad \frac{dE}{dt} = 0 \Rightarrow E = \text{const.}$$

$$\Rightarrow \text{write } \vec{p} = m\gamma\vec{v} = m\gamma c^2 \cdot \frac{\vec{v}}{c^2} = E\cdot\frac{\vec{v}}{c^2}$$

$$\Rightarrow \frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B} \Rightarrow \text{define } \boxed{\vec{\omega}_B = \frac{q\vec{B}c}{E} = \frac{q\vec{B}}{\gamma mc}}$$

(precession frequency)

$$\Rightarrow \frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B \quad \Rightarrow \text{if } \vec{B} = B\hat{z} \Rightarrow \vec{\omega}_B = \omega_B\hat{z}$$