Last time | **Relativistic Particles in Electromagnetic Fields.** (cont'd)

Lorentz force:
$$\begin{cases} \dfrac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right] \\ \dfrac{dE}{dt} = q\,\vec{v}\cdot\vec{E} \end{cases}$$

Summarized in relativistic 4-vector notation as

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c}\, u_\nu F^{\mu\nu}}$$

$u_\nu \sim$ 4-velocity, $F^{\mu\nu} \sim$ field strength tensor

We constructed the action for particle-field interactions

$$\boxed{S_{int} = -\frac{q}{c}\int_1^2 dx_\mu A^\mu}$$

and the Lagrangian:

$$\boxed{L = \underbrace{-mc^2\sqrt{1-\frac{v^2}{c^2}}}_{L_{free}} \underbrace{- q\Phi + \frac{q}{c}\vec{v}\cdot\vec{A}}_{L_{int}}}$$

Canonical momentum $\boxed{\vec{P} = \vec{p} + \frac{q}{c}\vec{A}}$.

Hamiltonian: $H = \vec{P} \cdot \vec{v} - L$

$$\Rightarrow \quad \boxed{H = \sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2} + q\,\Phi}$$

energy of a pt. particle in E&M field

$$\Rightarrow \boxed{H = \sqrt{m^2c^4 + (c\vec{P} - e\vec{A})^2} + e\Phi}$$ total energy of the particle

$$\text{“}\sqrt{m^2c^4 + p^2c^2}$$

Motion of a point charge in external $\vec{E}, \vec{B}$ fields:

We have Lorentz force $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$

and energy change $\frac{dE}{dt} = q\,\vec{v}\cdot\vec{E}$.

A. Constant Uniform Electric Field.

$$\frac{d\vec{p}}{dt} = q\vec{E} \Rightarrow \vec{p} = q\vec{E}t + \text{const} \Rightarrow \text{if}$$

the particle starts from rest $\Rightarrow \vec{p}\big|_{t=0} = 0$

$$\Rightarrow \vec{p} = q\vec{E}t \Rightarrow \text{is } \vec{E} = E\hat{x} \Rightarrow$$

$$\Rightarrow p_x = qEt, \quad p_y = p_z = 0$$

$$\Rightarrow \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = qEt \Rightarrow m^2\left(\frac{dx}{dt}\right)^2 = q^2E^2t^2\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t^2}}$$

$$\Rightarrow x(t) = \int_0^t dt' \frac{qEt'}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t'^2}} = \frac{\cancel{q}\cancel{E}}{\cancel{m}} \frac{c^2 \cancel{m}}{\cancel{q}^2 E^2}\left(\sqrt{1 + \frac{q^2E^2}{m^2c^2}t^2} - 1\right)$$

assume $X(0) = 0$

$$\Rightarrow x(t) = \frac{mc^2}{qE}\left(\sqrt{1 + \frac{q^2E^2}{m^2c^2}t^2} - 1\right)$$

$\Rightarrow$ as $t \to \infty \Rightarrow x(t) \approx c\ t$. ~ linear in t! moves with speed of light!

$\Rightarrow$ if c is large $\Rightarrow$ expand in powers of $\frac{1}{c} \Rightarrow$

$$\Rightarrow x(t) \approx \frac{1}{2} \frac{qE}{m} t^2 = \frac{1}{2} a t^2$$ ~ well-known classical NR result!

B. Constant Uniform Magnetic Field

$$\frac{d\vec{p}}{dt} = \frac{q}{c} \vec{v} \times \vec{B}, \quad \frac{dE}{dt} = 0 \Rightarrow E = \text{const.}$$

$$\Rightarrow \text{write } \vec{p} = m\gamma\vec{v} = m\gamma c^2 \cdot \frac{\vec{v}}{c^2} = E \cdot \frac{\vec{v}}{c^2}$$

$$\Rightarrow \frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v} \times \vec{B} \Rightarrow \text{define } \vec{\omega}_B = \frac{q\vec{B}c}{E} = \frac{q\vec{B}}{\gamma mc}$$

(precession frequency)

$$\Rightarrow \frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B \quad \Rightarrow \text{if } \vec{B} = B\hat{z} \Rightarrow \vec{\omega}_B = \omega_B \hat{z}$$

$\Rightarrow$ get $\dot{V}_x = \omega_B V_y$, $\dot{V}_y = -\omega_B V_x$, $\dot{V}_z = 0$

$\Rightarrow$ $\ddot{V}_x = \omega_B \dot{V}_y = -\omega_B^2 V_x \Rightarrow V_x = V_{0\perp} \cdot e^{\pm i\omega_B t}$

$\Rightarrow V_y = \frac{1}{\omega_B} \dot{V}_x = \pm i V_{0\perp} e^{\pm i\omega_B t}$

$\Rightarrow$ taking real parts write $V_x = V_{0\perp} \cos(\omega_B t + \alpha)$

$\Rightarrow V_y = -V_{0\perp} \sin(\omega_B t + \alpha) \Rightarrow \sqrt{V_x^2 + V_y^2} = V_{0\perp}$

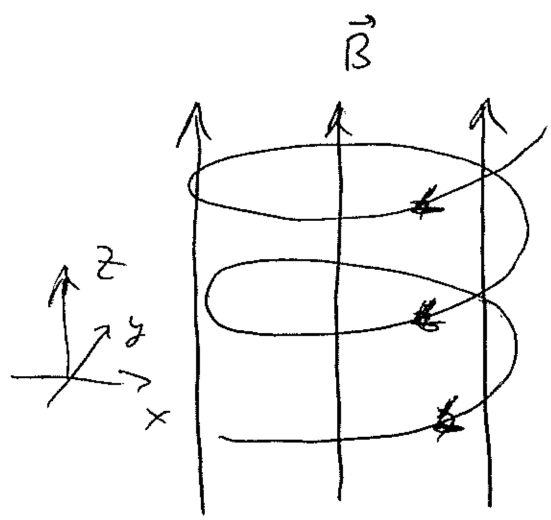
~ transverse (w.r.t. $\vec{B}$) velocity

$\Rightarrow$ as $V_x = \dot{x} = V_{0\perp} \cos(\omega_B t + \alpha) \Rightarrow$

$$\Rightarrow \begin{cases} x(t) = X_0 + r \sin(\omega_B t + \alpha) \\ y(t) = y_0 + r \cos(\omega_B t + \alpha) \\ z(t) = z_0 + V_{0z} t \end{cases}$$

$$r = \frac{V_{0\perp}}{\omega_B} = \frac{V_{0\perp} E}{qBc} = \frac{c p_{0\perp}}{qB}.$$

Motion of a positive charge is shown here →



C. Constant Uniform Electric and Magnetic Fields

if one has both $\vec{E}$ & $\vec{B} \neq 0$, $\vec{E} \perp \vec{B}$

(i) $B > E$ $\Rightarrow$ boost into a frame where $\vec{E}' = 0$ $\Rightarrow$ get motion in a uniform constant $\vec{B}'$-field $\Rightarrow$ back to case B. $\left(I_1 = 2(B^2 - E^2) > 0 \right)$

(ii) $E > B$ $\Rightarrow$ boost into a frame where $\vec{B}' = 0$ $\left(I_1 = 2(B^2 - E^2) < 0 \right)$ $\Rightarrow$ get motion in a uniform constant $\vec{E}'$-field $\Rightarrow$ back to case A.

(Note that since $\vec{E} \perp \vec{B}$ $\Rightarrow$ $I_2 = -4\vec{E}\cdot\vec{B} = 0$ $\Rightarrow$ there could be a frame where one of the fields is zero!)

Lagrangian for the Electromagnetic Field.

First let's discuss the differences between Lagrangians for fields vs. point particles:

for point particles $L = L(q_i, \dot{q}_i, t)$

and the action is $S = \int dt\, L(q_i, \dot{q}_i, t)$

$q_i \sim$ degrees of freedom (e.g. coordinates)

$\dot{q}_i = \frac{dq_i}{dt} \sim$ generalized velocities.

Suppose instead of discrete charges we'll have a field $\phi(\vec{x}, t)$ (e.g. wave-function for a particle in QM, or EM potential ...)

Classical Mechanics		Classical Field Theory
q_i	$\longrightarrow$	$\phi(\vec{x}, t)$
i	$\longrightarrow$	$\vec{x}, t$
$\dot{q}_i$	$\longrightarrow$	$\partial_\mu \phi(\vec{x}, t)$ $\mu = 0, 1, 2, 3$

$$L(q_i, \dot{q}_i, t) \rightarrow \int d^3x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

lagrangian density

Such that the action is

$$S = \int dt\, L = \int \underbrace{dt\, d^3x}_{\frac{1}{c} d^4x \leftarrow \text{Lorentz scalar}} \mathcal{L}(\phi, \partial_\mu \phi) =$$

as $d^4x' = \left|\det\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right| d^4x = d^4x$

$$= \frac{1}{c}\int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$\Rightarrow$ $\mathcal{L}$ is a Lorentz-scalar. (why?)

$$\Rightarrow \boxed{S = \frac{1}{c}\int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)}$$

Let's find the equations of motion: have to vary the action S w.r.t. $\phi \rightarrow \phi + \delta\phi$

$$\Rightarrow 0 = \delta S = \int \frac{d^4x}{c}\left[\frac{\partial \mathcal{L}}{\delta \phi}\delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right]$$

$\Rightarrow$ as $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ $\Rightarrow$ parts

$$0 = \int \frac{d^4x}{c}\left[\frac{\delta \mathcal{L}}{\delta \phi}\delta\phi - \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right)\delta\phi\right] +$$

$+ \underbrace{\text{surface term}}_{=0}.$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0.}$$ Euler-Lagrange equations for a field ϕ.

Now, let's find $\mathcal{L}$ for EM fields, $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu)$

$\Rightarrow$ EM field have superposition principle

~ equations of motion (Maxwell equ's) are linear $\Rightarrow$ $\mathcal{L}$ has to be quadratic in A_μ.

$\Rightarrow$ $\mathcal{L}$ is a Lorentz-scalar and is gauge-invariant ($A_\mu \to A_\mu + \partial_\mu \Lambda$) $\Rightarrow$ the only quadratic invariants we can build out of $F^{\mu\nu}$ & $\tilde{F}^{\mu\nu}$ are

$I_1 \propto F_{\mu\nu} F^{\mu\nu}$ $(= -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})$ and $I_2 \propto F_{\mu\nu}\tilde{F}^{\mu\nu}$.

But: I_2 is a pseudo-scalar under parity

$(I_2 \to -I_2$ if $\vec{x} \to -\vec{x}) \Rightarrow$ can't be in $\mathcal{L}$

(actually, I_2 can be written as $\partial_\mu K^\mu$, with K_μ some 4-vector $\Rightarrow \int d^4x\, I_2 = \int d^4x\, \partial_\mu K^\mu = \int_{\text{surface}} d\sigma_\mu K^\mu = 0$)

$\Rightarrow \mathcal{L} \propto F_{\mu\nu}F^{\mu\nu} \Rightarrow$ picking normalization to get Maxwell equs write $\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}}$

Remember the interaction action

$$S_{int} = -\frac{q}{c}\int dt \, \frac{1}{\gamma}\, u_\mu A^\mu = -\frac{1}{c}\int dt\, d^3x \sum_i q_i \frac{1}{\gamma_i} \cdot$$

$$\cdot\, u^i_\mu \; \delta^3(\vec{x}-\vec{x}_i)\, A^\mu(x)$$ for a set of discrete

charges $\{q_i\}$; $\left.\begin{array}{l} c\sum_i q_i\, \delta^3(\vec{x}-\vec{x}_i) \to c\rho(\vec{x}) \\ \sum_i q_i\, \vec{v}^i \delta^3(\vec{x}-\vec{x}_i) \to \vec{J}(\vec{x}) \end{array}\right\} J^\mu$

As $\frac{u^i_\mu}{\gamma_i} = (c, \vec{v}^i) \Rightarrow S_{int} = -\frac{1}{c^2}\int d^4x \; J_\mu A^\mu$

where $J^\mu = (c\rho, \vec{J})$. $\quad \left(S = \int d^4x \, \frac{1}{c}\, \mathcal{L} \right)$

$$\Rightarrow \boxed{\mathcal{L}_{int} = -\frac{1}{c}\, J_\mu A^\mu}$$

$\Rightarrow$ the full Lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

Its Euler-Lagrange equations should give Maxwell equations: start by rewriting

$$\mathcal{L} = -\frac{1}{16\pi}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{c} J_\mu A^\mu$$

Euler - Lagrange equations (equations of motion) for A_μ are

$$\boxed{\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0.}$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = -\frac{1}{c} J^\mu$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} \frac{\delta(F_{\alpha\beta} F^{\alpha\beta})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} g^{\alpha\rho} g^{\beta\sigma}.$$

$$\cdot \frac{\delta(F_{\alpha\beta} F_{\rho\sigma})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} g^{\alpha\rho} g^{\beta\sigma} \Big[\left(\delta_\alpha^\nu \delta_\beta^\mu - \delta_\alpha^\mu \delta_\beta^\nu \right) F_{\rho\sigma}$$

$$+ F_{\alpha\beta} \left(\delta_\rho^\nu \delta_\sigma^\mu - \delta_\rho^\mu \delta_\sigma^\nu \right) \Big] = -\frac{1}{16\pi} \left[F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right]$$

$$= \frac{1}{4\pi} F^{\mu\nu}$$

$\Rightarrow$ EOM are $\quad -\frac{1}{c} J^\mu - \frac{1}{4\pi} \partial_\nu F^{\mu\nu} = 0$

$\Rightarrow$ $\boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu}$ ~ Maxwell equations! (in Gaussian units)