Last time

Derived an expression for electrostatic energy in dielectrics:

$$\boxed{\delta W = \int d^3x \, \vec{E} \cdot \delta \vec{D}}$$

For linear & isotropic dielectrics:

$$\boxed{W = \frac{1}{2} \int d^3x \, \vec{E} \cdot \vec{D}}$$

Force:

$$\boxed{F_x = - \left(\frac{\partial W}{\partial x} \right)_Q}$$

$$\boxed{F_x = \left(\frac{\partial W}{\partial x} \right)_V}$$

note the positive sign!

Magnetostatics

$\Rightarrow$ main difference from electrostatics is due to absence of magnetic monopoles (no equivalent of point charges).

Instead one deals with magnetic dipoles:

loop of current carries magnetic dipole $\vec{m}$ $(m = I S)$

Torque on $\vec{m}$ is $\vec{N} = \vec{m} \times \vec{B}$, where

$\vec{B}$ is <u>magnetic induction</u> (aka magnetic flux density) OR magnetic field

(Analogy to electric dipoles: if we have dipole $\vec{p}$ in electric field $\vec{E}$:

$$W = -\vec{p} \cdot \vec{E} =$$
$$= -p E \cdot \cos\alpha$$

torque is

$$N = + \frac{\partial W}{\partial \alpha} = p E \sin\alpha \Rightarrow \vec{N} = \vec{p} \times \vec{E} \;)$$

<u>Conservation of Charge</u>

<u>Continuity</u>: if $\rho(\vec{x}, t)$ is charge density and $\vec{J}(\vec{x}, t)$ is current density

$$\left(\rho = \frac{\text{charge}}{\text{volume}}, \quad J = \frac{\text{charge} \cdot \text{velocity}}{\text{volume}} \right)$$

$$\frac{\text{charge} \cdot \text{velocity}}{\text{volume}} = \frac{\text{current}}{\text{area}}$$

$\Rightarrow$ the change in total charge in enclosed volume V should be equal to the amount of charge that flowed in/out of the volume. Hence:

$$\underbrace{\Delta Q}_{\text{as } Q=\int d^3x\, \rho(\vec{x},t)} = \int_V d^3x \left[\rho(\vec{x},t+\Delta t) - \rho(\vec{x},t)\right] = -\Delta t \oint_S da \cdot \underbrace{J_n}_{=\hat{n}\cdot\vec{J}}$$

$$= -\Delta t \int_V d^3x\ \vec{\nabla}\cdot\vec{J}$$

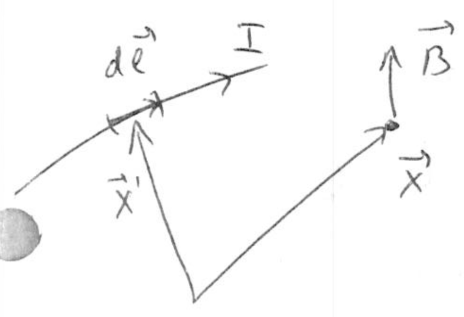
divergence theorem

$\Rightarrow$ as $\rho(\vec{x},t+\Delta t) - \rho(\vec{x},t) \approx \Delta t \cdot \frac{\partial \rho}{\partial t}(\vec{x},t)$

$$\Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

in the static case $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = 0}$.

Biot and Savart Law



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

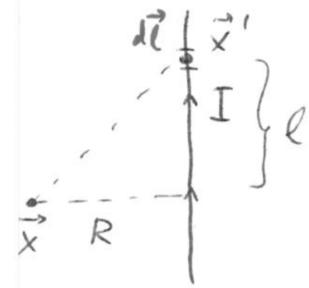
$\frac{\mu_0}{4\pi} = 10^{-7} \frac{\text{Newtons}}{\text{Ampere}^2}$, 1 Ampere = 1 Coulomb / 1 Second for a point charge q moving with velocity $\vec{v}$:

$$I\,d\vec{l} = q\,\vec{v} \Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\, q\, \frac{\vec{v}\times\vec{x}}{x^3}.$$

Transforming current I into current density $\vec{J}$ via $I\,d\vec{l} = \vec{J}\cdot d^3x$ we write

$$\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

This is magnetic induction due to any current density $\vec{J}(\vec{x})$.



Example: a wire carrying current I:

$$|\vec{B}| = \frac{\mu_0}{4\pi} I \cdot \int_{-\infty}^{\infty} \frac{dl}{l^2+R^2} \cdot \underbrace{\frac{R}{\sqrt{R^2+l^2}}}_{\text{sin of the angle between } d\vec{l}\ \&\ (\vec{x}-\vec{x}')} =$$

$$= \frac{\mu_0}{4\pi} I R \int_{-\infty}^{\infty} \frac{dl}{(l^2+R^2)^{3/2}} = \frac{\mu_0}{2\pi}\frac{I}{R}.$$

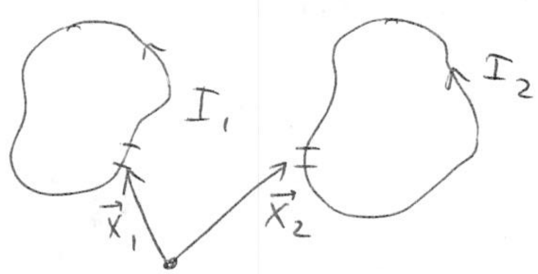
$$\int_{-\infty}^{\infty} \frac{dl}{(l^2+R^2)^{3/2}} = \left.\frac{l}{R^2\sqrt{R^2+l^2}}\right|_{-\infty}^{\infty} = \frac{2}{R^2}$$

Ampere's Law.

The force on a current element $I_1 \, d\vec{l}_1$ due to magnetic field $\vec{B}$ is $\boxed{d\vec{F} = I_1 \, d\vec{l}_1 \times \vec{B}}$

For a point charge q moving with velocity $\vec{v}$ write $\vec{F} = q \, \vec{V} \times \vec{B}$ (Lorentz force)

Imagine two loops of current: the force on loop #1 due to loop #2 is



$$\vec{F}_{12} = I_1 \int d\vec{l}_1 \times \vec{B}_2$$

Due to Biot & Savart law, $\vec{B}_2 = \frac{\mu_0}{4\pi} I_2 \int \frac{d\vec{l}_2 \times \vec{x}_{12}}{x_{12}^3}$

where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. Substituting:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3}$$

as
$$\frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3} = \frac{d\vec{l}_2 \, (d\vec{l}_1 \cdot \vec{x}_{12})}{x_{12}^3} - \vec{x}_{12} \, \frac{d\vec{l}_1 \cdot d\vec{l}_2}{x_{12}^3}$$

and, since $\vec{\nabla}_1 \frac{1}{|\vec{x}_{12}|} = - \frac{\vec{x}_{12}}{|x_{12}|^3}$, the first term vanishes and we write:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_{12}|^3} \vec{x}_{12}$$

attractive if $I_1 \parallel I_2$
repulsive if I_1 & I_2 anti-parallel

Ampere's law of force between two current loops.

As $I\, d\vec{l} = \vec{J}\, d^3x \Rightarrow$ for two localized current densities $\vec{F}_{12} = -\frac{\mu_0}{4\pi} \int d^3x_1\, d^3x_2 \frac{\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)}{|\vec{x}_{12}|^3} \vec{x}_{12}$.

For current density Ampere's law gives:

$$\boxed{\vec{F} = \int d^3x\, \vec{J}(\vec{x}) \times \vec{B}(\vec{x})}$$

The resulting torque is $\boxed{\vec{N} = \int d^3x\, \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}(\vec{x}))}$.

Differential Equations of Magnetostatics.

Start with Biot & Savart law:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = -\frac{\mu_0}{4\pi} \int d^3x' \cdot \vec{J}(\vec{x}') \times \vec{\nabla}_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = \frac{\mu_0}{4\pi} \vec{\nabla}_x \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$\Rightarrow$ we recast $\vec{B}$ as a curl of some vector field.

Definition Vector potential $\vec{A}$ is defined by

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

(Analogue of electrostatic potential Φ with $\vec{E} = -\vec{\nabla}\Phi$)

$\vec{A}$ is not observable directly (in classical physics)

$\vec{B}$ is observable

We have the freedom of redefining

$$\vec{A}(\vec{x}) \to \vec{A}(\vec{x}) + \vec{\nabla}\psi(\vec{x})$$

for any random scalar function $\psi(\vec{x})$:

as $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$, $\vec{B}$ does not change $\Rightarrow$

$\Rightarrow$ gauge invariance!

(cf. $\Phi \to \Phi + \text{const}$ in electrostatics)

$\vec{A}$ is defined up to a gradient.

Biot - Savart law gives us

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{\nabla}\psi$$