Last time: **Magnetostatics** (cont'd)

$\vec{N} = \vec{m} \times \vec{B}$ => defines magnetic field $\vec{B}$

$\vec{J}$ = current density ($I d\vec{\ell} \Rightarrow \vec{J}\, d^3x$)

conservation of charge:

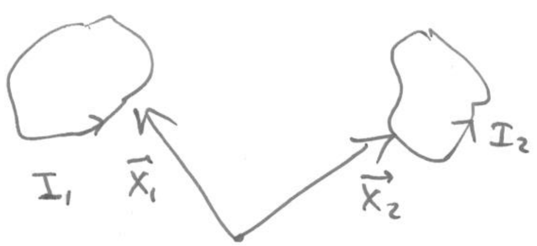
$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$ (continuity)

Biot & Savart Law: $$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$

Ampere's Law: $d\vec{F} = I_1\, d\vec{\ell}_1 \times \vec{B}$

$\vec{F} = q\,\vec{v} \times \vec{B}$ (Lorentz force)

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{x}_{12}|^3}\, \vec{x}_{12}$$



$\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$

Differential Equations of Magnetostatics

$$\vec{B} = \vec{\nabla}_x \times \left(\frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$$

Def. Vector potential $\vec{A}$: $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

$$\Rightarrow \boxed{\vec{A}_{(\vec{x})} = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{\nabla} \psi}$$

arbitrary function

⇓

gauge invariance!

as $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$ (analogy of $\vec{\nabla} \times \vec{E} = 0$)

$\Rightarrow$ no magnetic monopoles ~ no point sources of $\vec{B}$

On the other hand,

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} =$$

$$= \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \int d^3x' \vec{J}(\vec{x}') \cdot$$

$$\cdot \underbrace{\vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|}}_{-\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} \text{ \& do parts}} - \frac{\mu_0}{4\pi} \int d^3x' \vec{J}(\vec{x}') \underbrace{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}}_{= -4\pi \delta^3(\vec{x}-\vec{x}')} =$$

$$= \mu_0 \vec{J}(\vec{x}) + \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}_{= 0}$$

(continuity equation is steady state)

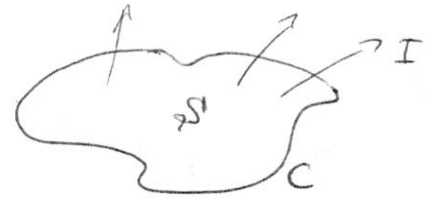
$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$ (analogy of $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$)

if $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \mu_0 \varepsilon_0 \vec{\nabla} \cdot \frac{-\partial}{\partial t} \vec{E} = +\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow$ ~ displacement current

$\Rightarrow$ get $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

To derive an analogy of Gauss's law, integrate

$$\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\, d\vec{l} \quad \left(\substack{\text{stokes's}\\ \text{thm}}\right)$$



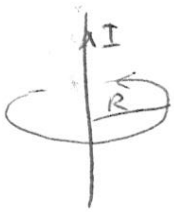
$$\parallel$$

$$\mu_0 \int_S da\ \hat{n}\cdot\vec{J} \Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S da\ \hat{n}\cdot\vec{J} = \mu_0 I}$$

Ampere's law

$I \sim$ total current through the loop.

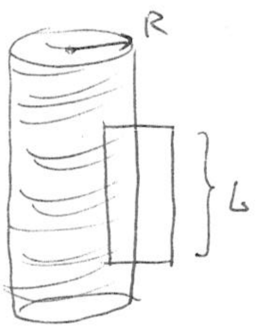
<u>Example</u>: find $\vec{B}$ of a straight wire carrying current I:



$$B\cdot 2\pi R = \mu_0 I \Rightarrow B = \frac{\mu_0}{2\pi}\frac{I}{R}$$

(cf. with what we found using Biot-Savart law earlier)

<u>Example</u>: infinite solenoid, N coils per unit length:



$$B_{in}\cdot L = \mu_0 I\cdot N\cdot L \Rightarrow B_{in} = \mu_0 I N$$

uniform magnetic field inside!

$B_{out} = 0$.

Finally, we know that $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ & $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$

outside: $B_\varphi = 0$, $B_r = 0$

$\Box \Rightarrow B(r_1) = B$ [illegible]

Writing $\vec{B} = \vec{\nabla} \times \vec{A}$ automatically satisfies the first equation. The second yields:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}}$$

As we have gauge freedom $\vec{A} \to \vec{A} + \vec{\nabla} \psi$,

$\Rightarrow$ can demand that in new gauge $\vec{\nabla} \cdot \vec{A}_{new} = 0$

$$\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla} \cdot \psi \Rightarrow \vec{\nabla} \cdot \vec{A}_{new} = \vec{\nabla} \cdot \vec{A}_{old} + \vec{\nabla}^2 \psi = 0$$

$\Rightarrow \nabla^2 \psi = -\vec{\nabla} \cdot \vec{A}_{old}$ $\Rightarrow$ can always find ψ by solving this Poisson-like equation

$\vec{\nabla} \cdot \vec{A} = 0$ ~ Coulomb gauge condition.

in Coulomb gauge $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

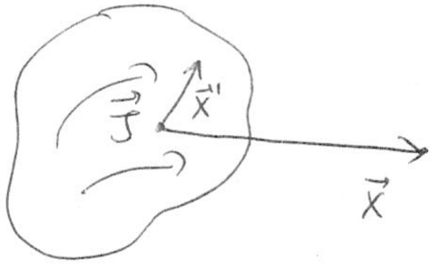
$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (\psi = \text{const}).$$

or any ftn satisfying $\nabla^2 \psi = 0$.

Magnetic Fields of a Localized Current Distribution:

Magnetic Moment.

Imagine a localized current distribution:



We need to find vector potential far away from the currents: start with

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ to properly expand $\vec{A}(\vec{x})$ in powers of $\frac{1}{r}$ we need vector spherical harmonics ~ we'll maybe talk about them next quarter.

$\Rightarrow$ Instead expand

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots$$

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\int d^3x' \, J_i(\vec{x}') + \frac{\mu_0}{4\pi}\frac{\vec{x}\cdot}{|\vec{x}|^3}\int d^3x' .$$

$$\cdot \vec{x}' \, J_i(\vec{x}') + \ldots$$

Now, $\int d^3x' \, J_i(\vec{x}') = \int d^3x' \left[\vec{\nabla}'\cdot\left(x_i' \, \vec{J}(\vec{x}')\right) - x_i' \, \vec{\nabla}\cdot\vec{J}\right]$