Last time: Derived

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \quad \text{and} \quad \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Ampere's Law

for magnetostatics:

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \quad \text{and} \quad \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$$

$$\hookrightarrow \boxed{\oint_C d\vec{\ell}\cdot\vec{B} = \mu_0 I}$$ integral form

$B = \vec{\nabla}\times\vec{A}$ => can choose a gauge such that

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$ ($\vec{\nabla}\cdot\vec{A} = 0$ Coulomb gauge)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Writing $\vec{B} = \vec{\nabla} \times \vec{A}$ automatically satisfies the first equation. The second yields:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}}$$

As we have gauge freedom $\vec{A} \to \vec{A} + \vec{\nabla} \psi$,

$\Rightarrow$ can demand that in new gauge $\vec{\nabla} \cdot \vec{A}_{new} = 0$

$$\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla} \cdot \psi \Rightarrow \vec{\nabla} \cdot \vec{A}_{new} = \vec{\nabla} \cdot \vec{A}_{old} + \vec{\nabla}^2 \psi = 0$$

$\Rightarrow \nabla^2 \psi = - \vec{\nabla} \cdot \vec{A}_{old}$ $\Rightarrow$ can alway find ψ by solving this Poisson-like equation

$\vec{\nabla} \cdot \vec{A} = 0$ ~ Coulomb gauge condition.

in Coulomb gauge $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

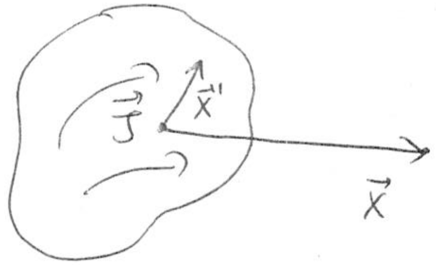
$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (\psi = \text{const}).$$

or any ftn satisfying $\nabla^2 \psi = 0$.

Magnetic Fields of a Localized Current Distribution:

Magnetic Moment.

Imagine a localized current distribution:



We need to find vector potential far away from the currents: start with

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ to properly expand $\vec{A}(\vec{x})$ in powers of $\frac{1}{r}$ we need vector spherical harmonics ~ we'll maybe talk about them next quarter.

$\Rightarrow$ Instead expand

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots$$

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\int d^3x' \, J_i(\vec{x}') + \frac{\mu_0}{4\pi}\frac{\vec{x}\cdot}{|\vec{x}|^3}\int d^3x' .$$

$$\cdot\vec{x}' \, J_i(\vec{x}') + \ldots$$

Now, $\int d^3x' \, J_i(\vec{x}') = \int d^3x' \left[\vec{\nabla}'\cdot\left(x_i' \, \vec{J}(\vec{x}')\right) - x_i' \, \vec{\nabla}\cdot\vec{J}\right],$

First term becomes a surface integral

$$\int_S da\ x_i J_n = 0 \quad \text{as current is localized}$$

Second term is also $\emptyset$ as $\vec{\nabla}\cdot\vec{J} = 0$ (continuity)

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}\cdot\int d^3x'\,\vec{x}'\,J_i(\vec{x}')$$

as $\vec{\nabla}\cdot\vec{J} = 0$.

Now, $0 = \int d^3x'\,\vec{\nabla}'\cdot\left(x_i' x_j'\,\vec{J}(\vec{x}')\right) = \int d^3x'\left[x_i' J_j + \right.$

$\left. + x_j' J_i\right] \Rightarrow \int d^3x'\left[x_i' J_j + x_j' J_i\right] = 0$

$$\Rightarrow \vec{x}\cdot\int d^3x'\,\vec{x}'\cdot J_i(\vec{x}') \equiv \sum_j x_j \int d^3x'\,x_j' J_i =$$

$$= -\frac{1}{2}\sum_j x_j \int d^3x'\left[x_i' J_j - x_j' J_i\right] =$$

$$= -\frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\,x_j \int d^3x'\,(\vec{x}'\times\vec{J})_k$$

as $(\vec{x}'\times\vec{J})_k = \varepsilon_{ki'j'}\,x_{i'}'\,J_{j'}$ and

$\varepsilon_{ijk}\,\varepsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}$.

Finally we obtain $-\frac{1}{2}\left[\vec{x}\times\int d^3x'\,(\vec{x}'\times\vec{J})\right]_i$, Such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left(-\frac{1}{2}\right)\frac{\vec{x}}{|\vec{x}|^3} \times \int d^3x' \, \vec{x}' \times \vec{J}$$

Definition. Defining <u>magnetic moment</u>

$$\vec{m} = \frac{1}{2}\int d^3x' \;\vec{x}' \times \vec{J}(\vec{x}')$$

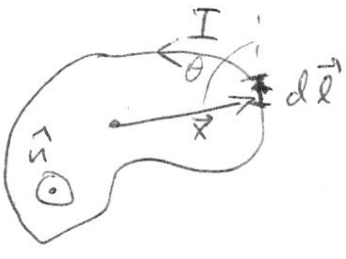
We obtain $\boxed{\vec{A}(\vec{x}) = \dfrac{\mu_0}{4\pi}\,\dfrac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\,\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}}$$ (cf. with $\vec{E}$ of a dipole)

Definition $\vec{M} = \frac{1}{2}\vec{x} \times \vec{J}(\vec{x})$ is the magnetic moment density, or, <u>magnetization</u>.

(More precisely, $\vec{B}(\vec{x}) = \dfrac{\mu_0}{4\pi}\left[\dfrac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \dfrac{8\pi}{3}\vec{m}\,\delta^3(\vec{x})\right]$)

Suppose the current is confined to a plane:



$$\vec{m} = \frac{1}{2} I \int \vec{x} \times d\vec{l}$$

$$\Rightarrow \text{as } |\vec{x} \times d\vec{l}| \cdot \frac{1}{2} = \frac{1}{2} x \, dl \cdot \sin\theta = \underbrace{da}_{\text{area element}}$$

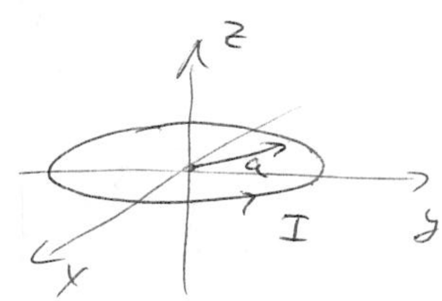
$\Rightarrow \left| \frac{1}{2} \int \vec{x} \times d\vec{\ell} \right| = S$ (area of the loop)

$\Rightarrow |\vec{m}| = I \cdot S$, or $\boxed{\vec{m} = I S \hat{n}}$

$\hat{n}$ is pointing out of the plane

$\vec{m}$ is independent of origin. Can you prove that?

Example: current loop:



$\Rightarrow \vec{m} = I \cdot \pi a^2 \cdot \hat{n} = I \pi a^2 \hat{z}$.

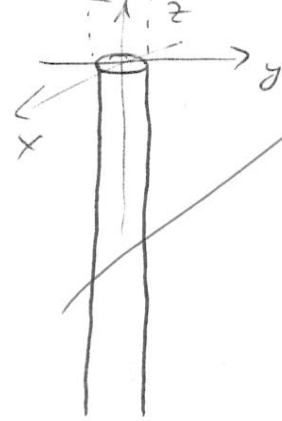
far from the loop.

$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0 I a^2}{4}$.

$\cdot \frac{\hat{z} \times \vec{x}}{\vec{x}^3} \Rightarrow$ in spherical coordinates

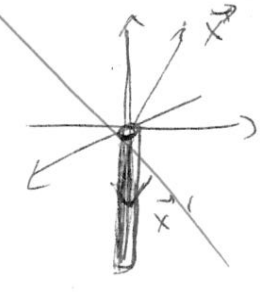
$$A_\varphi = \frac{\mu_0 I a^2}{4} \frac{\sin\theta}{r^2} , \quad A_\theta = A_r = 0.$$

Example consider a half-infinite ideal solenoid; it has current I and N loops per unit length. Each loop carries magn. moment



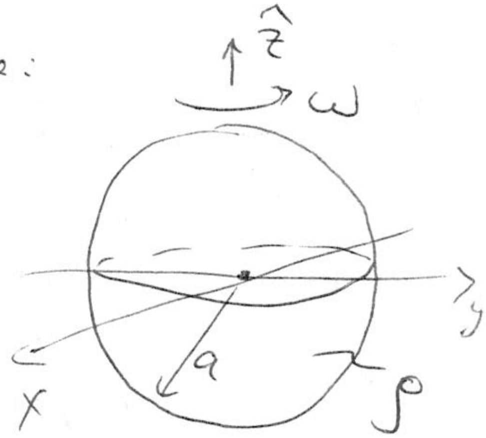
$\Delta \vec{m} = I \pi a^2 \hat{z}$ Assume that a is tiny $\Rightarrow$

$$\Rightarrow \vec{A}_{(\vec{x})} \simeq \frac{\mu_0}{4\pi} \int \frac{d\vec{m}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$



where $dm = I \pi a^2 N dz$.

Example: find magnetic dipole moment of a rotating uniformly charged sphere:



$$\vec{m} = \frac{1}{2}\int d^3x \;\; \vec{x}\times\vec{J}$$

$$\vec{J} = \rho\cdot\vec{v} = \rho\;\vec{\omega}\times\vec{x}$$

$$\Rightarrow \vec{m} = \frac{\rho}{2}\int d^3x \;\; \vec{x}\times(\vec{\omega}\times\vec{x}) =$$

$$= \frac{\rho}{2}\int d^3x \left[\vec{\omega}\,|\vec{x}|^2 - \vec{x}\,(\vec{x}\cdot\vec{\omega})\right]$$

$$\Rightarrow \text{as } \vec{\omega} = \omega\,\hat{z} \Rightarrow m_x = m_y = 0$$

$$\Rightarrow m_z = \frac{\rho}{2}\,\omega\int d^3x\left[r^2 - z^2\right] =$$

$$= \frac{\rho}{2}\,\omega\cdot 2\pi\int_0^a dr\cdot r^2\int_{-1}^{1} d\cos\theta\left[r^2 - r^2\cos^2\theta\right] =$$

$$= \frac{\rho}{\cancel{2}}\,\omega\cdot\cancel{2}\pi\;\frac{a^5}{5}\left[2-\frac{2}{3}\right] = \pi\,\omega\,\rho\,a^5\cdot\frac{4}{15}$$

$$\Rightarrow \text{as } q = \frac{4}{3}\pi a^3\rho \Rightarrow \boxed{m = \frac{1}{5}\,q\,\omega\,a^2}.$$

Torque on $\vec{m}$: $\boxed{\vec{N} = \vec{m}\times\vec{B}(0)}$ (by definition of $\vec{B}$..)