Last time | Magnetic Fields of a Localized Current Distribution

Magnetic moment

$$\boxed{\vec{m} \equiv \frac{1}{2}\int d^3x \; \vec{x} \times \vec{J}(\vec{x})}$$ magnetic dipole moment

$$\boxed{\vec{M}(\vec{x}) = \frac{1}{2}\,\vec{x} \times \vec{J}(\vec{x})}$$ magnetization

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$ vector potential due to a dipole

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\,\frac{3\hat{n}(\vec{m}\cdot\hat{n}) - \vec{m}}{|\vec{x}|^3}}, \quad \hat{n} = \hat{r}.$$

planar loop:

$$\vec{m} = I S \hat{n}.$$

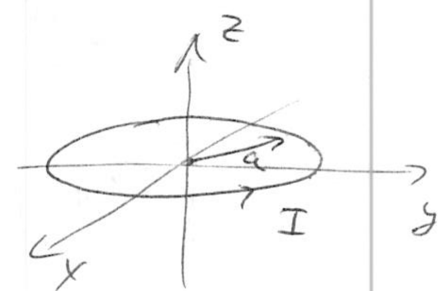
$$\Rightarrow \left| \frac{1}{2} \int \vec{x} \times d\vec{l} \right| = S \quad \text{(area of the loop)}$$

$$\Rightarrow |\vec{m}| = I \cdot S \quad , \text{or} \quad \boxed{\vec{m} = I S \hat{n}}$$

$\hat{n}$ is pointing out of the plane

$\vec{m}$ is independent of origin. Can you prove that?

<u>Example</u>: current loop:



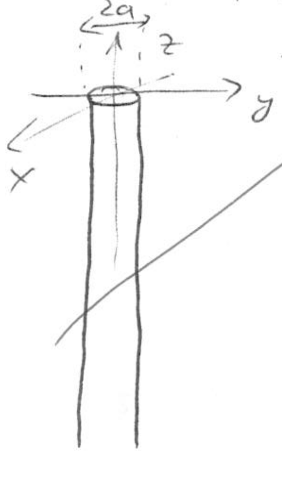
$$\Rightarrow \vec{m} = I \cdot \pi a^2 \cdot \hat{n} = I \pi a^2 \hat{z}.$$

far from the loop.

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0 I a^2}{4} \cdot \frac{\hat{z} \times \vec{x}}{\vec{x}^3} \Rightarrow \text{in spherical coordinates}$$

$$A_\varphi = \frac{\mu_0 I a^2}{4} \frac{\sin\theta}{r^2} \quad , \quad A_\theta = A_r = 0.$$

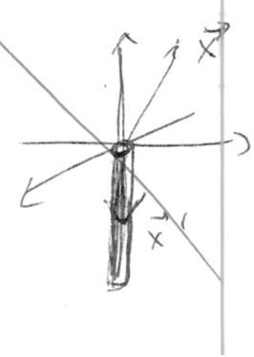
~~<u>Example</u> consider a half-infinite ideal solenoid; it has current I and N loops per unit length. Each loop carries magn. moment~~



$$\Delta \vec{m} = I \pi a^2 \hat{z}$$

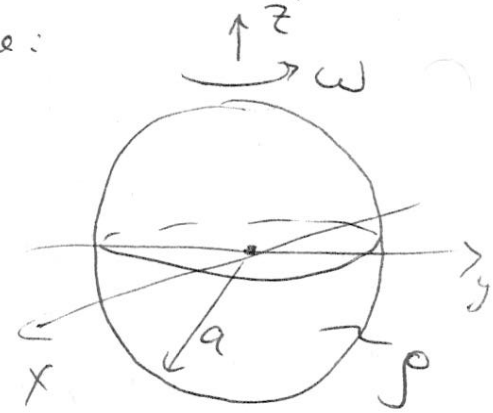
~~Assume that a is tiny~~ $\Rightarrow$

$$\Rightarrow \vec{A}_{(\vec{x})} \simeq \frac{\mu_0}{4\pi} \int \frac{d\vec{m}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$



~~where~~ $dm = I \pi a^2 N dz$

Example: find magnetic dipole moment of a rotating uniformly charged sphere:



$$\vec{m} = \frac{1}{2}\int d^3x \;\; \vec{x}\times\vec{J}$$

$$\vec{J} = \rho\cdot\vec{v} = \rho\;\vec{\omega}\times\vec{x}$$

$$\Rightarrow \vec{m} = \frac{\rho}{2}\int d^3x \;\; \vec{x}\times(\vec{\omega}\times\vec{x}) =$$

$$= \frac{\rho}{2}\int d^3x \left[\vec{\omega}\,|\vec{x}|^2 - \vec{x}\cdot(\vec{x}\cdot\vec{\omega})\right]$$

$$\Rightarrow \text{as } \vec{\omega} = \omega\,\hat{z} \Rightarrow m_x = m_y = 0$$

$$\Rightarrow m_z = \frac{\rho}{2}\,\omega\int d^3x\left[r^2 - z^2\right] =$$

$$= \frac{\rho}{2}\,\omega\cdot 2\pi\int_0^a dr\cdot r^2\int_{-1}^{1} d\cos\theta\left[r^2 - r^2\cos^2\theta\right] =$$

$$= \frac{\rho}{\cancel{2}}\,\omega\cdot\cancel{2}\pi\;\frac{a^5}{5}\left[2 - \frac{2}{3}\right] = \pi\,\omega\,\rho\,a^5\cdot\frac{4}{15}$$

$$\Rightarrow \text{as } q = \frac{4}{3}\pi a^3\rho \Rightarrow \boxed{m = \frac{1}{5}\,q\,\omega\,a^2}$$

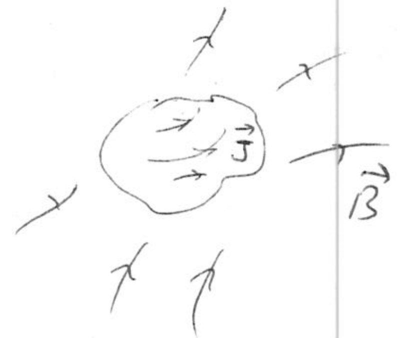
Torque on $\vec{m}$:

$$\boxed{\vec{N} = \vec{m}\times\vec{B}(0)}$$

(by definition of $\vec{B}$.)

Force on and Energy of a Localized Current.

Consider a system of localized currents in external magnetic induction $\vec{B}$:



$$\vec{F} = \int d^3x \; \vec{J}(\vec{x}) \times \vec{B}(\vec{x})$$

If $\vec{B}$ is slowly varying, write

$$\vec{B}(\vec{x}) = \vec{B}(0) + (\vec{x} \cdot \vec{\nabla})\, \vec{B}(0) + \ldots$$

$$\Rightarrow \vec{F} = \cancelto{0}{\left[\int d^3x \; \vec{J}(\vec{x})\right]} \times \vec{B}(0) + \int d^3x \; \vec{J}(\vec{x}) \times (\vec{x} \cdot \vec{\nabla})\, \vec{B}(0)$$

$$\Rightarrow F_i = \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \cdot (\vec{x} \cdot \vec{\nabla})\, B_k(0) =$$

$$= \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \; x_l \left(\partial_l B_k\right)\Big|_{\vec{x}=0} =$$

$$= \left(\partial_l B_k\right)\Big|_{\vec{x}=0} \varepsilon_{ijk} \int d^3x \; x_l J_j$$

$$\Rightarrow \text{as} \int d^3x \left(x_i J_j + x_j J_i\right) = 0 \quad \Rightarrow$$

$$F_i = \left(\partial_l B_k\right)\Big|_{\vec{x}=0} \varepsilon_{ijk} \underbrace{\frac{1}{2} \int d^3x \left[x_l J_j - x_j J_l\right]}_{\varepsilon_{ljn} \, m_n}$$

$$\left(\text{as } m_i = \frac{1}{2}\varepsilon_{ijk}\int d^3x\, x_j J_k \quad \Rightarrow\right.$$

$$\Rightarrow \varepsilon_{ljn}\cdot m_n = \frac{1}{2}\underbrace{\varepsilon_{ljn}\varepsilon_{nj'k}}_{\delta_{lj'}\delta_{jk} - \delta_{lk}\delta_{jj'}}\int d^3x\, x_{j'} J_k =$$

$$\left. = \frac{1}{2}\int d^3x \left[x_l J_j - x_j J_l\right]\right)$$

$$\Rightarrow F_i = \left(\partial_l B_k\right)\Big|_{\vec{x}=0} \varepsilon_{ijk}\varepsilon_{ljn} m_n =$$

$$= \left(\partial_l B_k\right)\Big|_{\vec{x}=0} m_n \cdot \left[\delta_{il}\delta_{kn} - \delta_{in}\delta_{kl}\right] =$$

$$= m_k\left(\partial_i B_k\right)\Big|_{\vec{x}=0} - m_i \underbrace{\left(\partial_k B_k\right)\Big|_{\vec{x}=0}}_{= 0 \text{ as } \vec{\nabla}\cdot\vec{B} = 0}$$

$$\Rightarrow \boxed{\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)}$$

If $\vec{F} = -\vec{\nabla} U$, with U the potential energy, $\Rightarrow \boxed{U = -\vec{m}\cdot\vec{B}}$

tends to align dipoles with the magnetic induction $\vec{B}$.