

Last time: Force & Energy of a Localized Current

$$\vec{F} = \nabla (\vec{m} \cdot \vec{B}) \quad \sim \text{force}$$

$$U = -\vec{m} \cdot \vec{B} \quad \sim \text{energy}$$

$$\vec{N} = \vec{m} \times \vec{B} \quad \sim \text{torque}$$



# Macroscopic Equations of Magnetostatics.

Similar to electrostatics, let's divide the current into "free" and "bound".

Bound currents are due to magnetic moments of molecules/atoms in the medium and are described by magnetization  $\vec{M}(\vec{x}) = \sum_i N_i \langle m_i \rangle$  magn. dip moment of type i molecule

The resulting vector-potential is  $\vec{M}(\vec{x}) = \sum_i m_i \delta(\vec{x} - \vec{x}_i)$  # molecules of type i per unit volume

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \underbrace{\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{\text{free current}} + \underbrace{\frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}_{\text{magnetization / bound currents}} \right\}$$

The 2nd term is

$$\int d^3x' \frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \int d^3x' \vec{M}(\vec{x}') \times \nabla' \frac{1}{|\vec{x} - \vec{x}'|} = -\nabla' \times \left[ \vec{M} \frac{1}{|\vec{x} - \vec{x}'|} \right] + (\nabla' \times \vec{M}) \frac{1}{|\vec{x} - \vec{x}'|}$$

(parts)  $\int d^3x \nabla \times \vec{A} = \int d^3x \nabla \times \vec{A}$

such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') + \nabla' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$\Rightarrow$  effective current density due to  $\vec{M}$  is

$$\vec{J}_M = \nabla \times \vec{M}$$

$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \left( \vec{\nabla} \times \vec{B} = \mu_0 \left[ \vec{J} + \vec{\nabla} \times \vec{M} \right] \right)$$

Define magnetic field  $\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}$

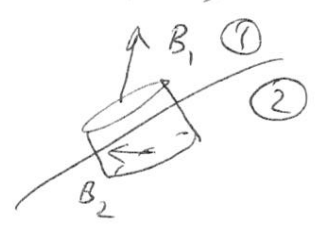
$\Rightarrow$  macroscopic equations of magnetostatics

are  $\vec{\nabla} \times \vec{H} = \vec{J}$  and  $\vec{\nabla} \cdot \vec{B} = 0$

(cf.  $\vec{\nabla} \cdot \vec{D} = \rho_f$ ,  $\vec{\nabla} \times \vec{E} = 0$  in electrostatics)

Boundary conditions:  $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$

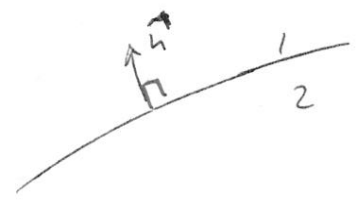
$$B_{1n} = B_{2n}$$



$(\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot \hat{t} dl$ , where  $\vec{K}$  = surface current density (amps/m<sup>2</sup>).  
*Stokes's thm*

$$\Rightarrow \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}$$

see next page



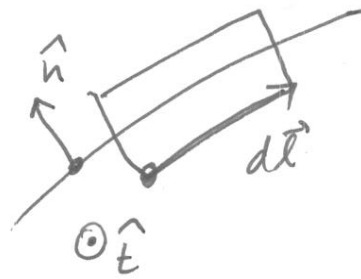
at the boundary of two media with magnetizations

$\vec{M}_1$  &  $\vec{M}_2$  :  $\vec{J}_m = \vec{\nabla} \times \vec{M}$  and  $\hat{n} \times (\vec{H}_1 - \vec{H}_2) =$   
 $= \hat{n} \times \left[ \frac{1}{\mu_0} \vec{B}_1 - \vec{M}_1 - \frac{1}{\mu_0} \vec{B}_2 + \vec{M}_2 \right] = \vec{K}$

$$\text{as } \hat{t} = \frac{d\vec{l} \times \hat{n}}{dl}$$

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$$\Rightarrow \hat{t} dl = d\vec{l} \times \hat{n}$$



$$\Rightarrow (\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot \hat{t} dl$$

$$\Rightarrow (\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot (d\vec{l} \times \hat{n}) = d\vec{l} \cdot (\hat{n} \times \vec{K})$$

as this is true for any  $d\vec{l}$  along the boundary,

$$(\vec{H}_2 - \vec{H}_1)_t = \hat{n} \times \vec{K}$$

$$\Rightarrow \hat{n} \times (\vec{H}_2 - \vec{H}_1)_t = \hat{n} \times (\hat{n} \times \vec{K})$$

$$\hat{n} \times (\hat{n} \times \vec{K}) = \hat{n} (\underbrace{\hat{n} \cdot \vec{K}}_{=0 \text{ as } \hat{n} \perp \vec{K}}) - \vec{K}$$

$$\Rightarrow \hat{n} \times (\vec{H}_2 - \vec{H}_1) = -\vec{K}, \text{ or}$$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K} \text{ as desired}$$



Therefore  $\frac{1}{\mu_0} \hat{n} \times (\vec{B}_1 - \vec{B}_2) = \vec{K} + \vec{K}_M$

where  $\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)$  ~ surface current due to magnetization.

What's missing? Relation between  $\vec{B}$  &  $\vec{H}$ !

For linear isotropic homogeneous media

$\vec{B} = \mu \vec{H}$ ,  $\mu$  is magnetic permeability.

As  $\vec{B} = \mu_0 (\vec{H} + \vec{M}) \Rightarrow \vec{M} = \frac{1}{\mu_0} \vec{B} - \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right) \vec{H} =$

$= \chi_m \vec{H} \Rightarrow \vec{M} = \chi_m \vec{H}$ ,  $\chi_m = \frac{\mu}{\mu_0} - 1$

magnetic susceptibility

$\chi_m > 0$  (~ paramagnetic (atoms/molecules have some angular momentum))  
( $\mu > \mu_0$ )

$\chi_m < 0$  (~ diamagnetic (no net angular momentum or atoms/molecules resist magnetic field))  
( $\mu < \mu_0$ )

$\vec{M} \neq 0$  "independent" of  $\vec{H}$  ~ ferromagnetic  
(even for  $\vec{H} = 0$ )

(paramagnetic materials with self-interactions between spins ~ e.g. Ising model)

(hard magnetic materials)

# Solving Boundary-Value Problems in Magnetostatics. (36)

① No ferromagnetics (no frozen  $\vec{M} \neq 0$  for all  $\vec{H}$ )

A. Vector Potential ( $\vec{J} \neq 0$ )

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \text{in Coulomb gauge } \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$\Rightarrow$  given  $\vec{J}$  can always find  $\vec{A}$ . (vacuum)

$$\text{As } \vec{\nabla} \times \vec{H} = \vec{J} = \vec{\nabla} \times \left( \frac{\vec{B}}{\mu} \right) = -\frac{1}{\mu} \nabla^2 \vec{A} \Rightarrow$$

$$\text{in medium } \nabla^2 \vec{A} = -\mu \vec{J}$$

B.  $\vec{J} = 0 \Rightarrow$  Magnetic Scalar Potential.

$$\vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\Rightarrow \text{write } \vec{H} = -\vec{\nabla} \Phi_M \Rightarrow \vec{B} = \mu \vec{H} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\text{gives } \vec{\nabla} \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Phi_M = 0 \sim \text{Laplace eqn.}$$

② Ferromagnetics ( $\vec{M} \neq 0, \vec{J} = 0$ )

A. Vector potential.

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \text{as } \vec{\nabla} \times \vec{H} = 0 \Rightarrow$$

$$\Rightarrow \vec{\nabla} \times \left( \frac{1}{\mu_0} \vec{B} - \vec{M} \right) = 0 \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{\nabla} \times \vec{M}$$