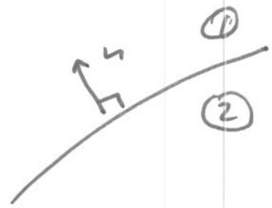
Last time: Macroscopic Equations of Magnetostatics

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\overbrace{\vec{J}(\vec{x}')}^{\text{free current dens.}} + \overbrace{\vec{\nabla}' \times \vec{M}(\vec{x}')}^{\text{magn. current density}}}{|\vec{x}-\vec{x}'|}}$$

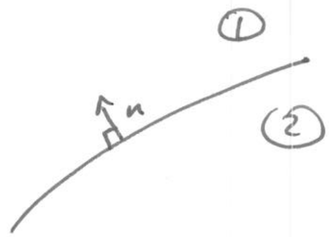
Def. $\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$

$\Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

boundary matching:



$\boxed{B_{1n} = B_{2n}}$



$\boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$

surface current density

$\boxed{\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)}$ ~ magnetization surface current

LIH medium $\boxed{\vec{B} = \mu \vec{H}}$, $\boxed{\vec{M} = \chi_m \vec{H}}$, $\chi_m = \frac{\mu}{\mu_0} - 1$

μ: magn. permeability; χ_m: magnetic susceptibility

$\chi_m > 0$ paramagnets, $\chi_m < 0$ diamagnets, $\vec{M} \neq 0$ ferromagnets

Therefore $\frac{1}{\mu_0}\hat{n}\times(\vec{B}_1-\vec{B}_2)=\vec{K}+\vec{K}_M$

where $\vec{K}_M=\hat{n}\times(\vec{M}_1-\vec{M}_2)$ ~ surface current due to magnetization.

What's missing? Relation between $\vec{B}$ & $\vec{H}$!

For linear isotropic homogeneous media

$\boxed{\vec{B}=\mu\vec{H}}$, μ is magnetic permeability.

As $\vec{B}=\mu_0(\vec{H}+\vec{M}) \Rightarrow \vec{M}=\frac{1}{\mu_0}\vec{B}-\vec{H}=\left(\frac{\mu}{\mu_0}-1\right)\vec{H}=$

$=\chi_m\vec{H} \Rightarrow \vec{M}=\chi_m\vec{H}$, $\chi_m=\frac{\mu}{\mu_0}-1$

magnetic susceptibility

$\chi_m>0$ $(\mu>\mu_0)$ ~ paramagnetic (atoms/molecules have some angular momentum)

$\chi_m<0$ $(\mu<\mu_0)$ ~ diamagnetic (no net angular momentum on atoms/molecules ~ resist magnetic field)

$\vec{M}\neq 0$ "independent" of $\vec{H}$ (even for $\vec{H}=0$) ~ ferromagnetic (paramagnetic materials with self-interactions between spins ~ e.g. Ising model)

(hard magnetic materials)

Solving Boundary-Value Problems in Magnetostatics

(I) No ferromagnetics (no frozen $\vec{M} \neq 0$ for all $\vec{H}$)

A. Vector Potential ($\vec{J} \neq 0$)

$\vec{B} = \vec{\nabla} \times \vec{A}$ => in Coulomb gauge $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

=> given $\vec{J}$ can always find $\vec{A}$. (vacuum)

As $\vec{\nabla} \times \vec{H} = \vec{J} = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu}\right) = -\frac{1}{\mu} \nabla^2 \vec{A}$ =>

in medium $\nabla^2 \vec{A} = -\mu \vec{J}$.

B. $\vec{J} = 0$ => Magnetic Scalar Potential.

$\vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0 \; , \; \vec{\nabla} \cdot \vec{B} = 0$

=> write $\vec{H} = -\vec{\nabla} \Phi_M \Rightarrow \vec{B} = \mu \vec{H} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

gives $\vec{\nabla} \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Phi_M = 0$ ~ Laplace eqn.

(II) Ferromagnetics ($\vec{M} \neq 0$, $\vec{J} = 0$) (non-zero $\vec{H}$, $\vec{B}$ may be due to boundary conditions)

A. Vector potential.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$ as $\vec{\nabla} \times \vec{H} = 0 \Rightarrow$

$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = 0 \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{\nabla} \times \vec{M}$

⇒ without "special" treatment of boundaries:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}}$$

← if $\vec{J} \neq 0$ just add $\vec{J}$ in the numerator

with "special" treatment of boundaries:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\mu_0}{4\pi}\oint_S da' \, \frac{\vec{M}(\vec{x}') \times \hat{n}'}{|\vec{x} - \vec{x}'|}}$$

(Remember ~ $\vec{K} = (\vec{M}_2 - \vec{M}_1) \times \hat{n}$ is the surface current)

B. Magnetic Scalar Potential

$0 = \vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0$ ⇒ defining $\vec{H} = -\vec{\nabla}\Phi_M$

we get $\nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ ~ Poisson-like eqn.

$$\Rightarrow \quad \boxed{\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}}$$

no explicit boundaries term (boundaries are included in careful eval. of $\vec{\nabla} \cdot \vec{M}$)

With explicit boundaries' term

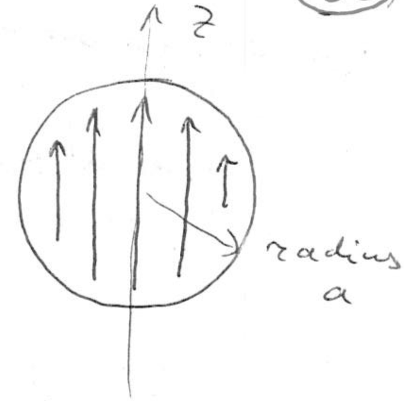
$$\boxed{\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi}\oint_S da' \, \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}}$$

"magnetic surface-charge" density

$$\boxed{\sigma_M = \hat{n} \cdot \vec{M}.}$$

Example: Ferromagnetic sphere:

$\vec{M}$ constant, $\vec{J}=0$, $\vec{M}=M\cdot\hat{z}$



Find $\vec{B}$, $\vec{H}$ everywhere.

$\Rightarrow$ Use magnetic potential:

$$\vec{\nabla}\cdot\vec{M}=0 \Rightarrow \Phi_M = \frac{1}{4\pi}\oint_S da' \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{M}{4\pi}\cdot a^2 \cdot \int d\Omega' \frac{\cos\theta'}{|\vec{x}-\vec{x}'|}$$

Using $\displaystyle \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi)$

and $Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$ we get

$$\Phi_M = \frac{Ma^2}{\cancel{4\pi}} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \, \cancel{4\pi} \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \cdot$$

$$\cdot Y_{lm}^*(\theta',\varphi') \cdot \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) = \left(\begin{array}{l}\text{only } l=1\\ m=0 \text{ is} \neq 0\end{array}\right)$$

$$= Ma^2 \frac{1}{3} \frac{r_<}{r_>^2} \cdot \cos\theta, \text{ where } r_{\gtrless} = \genfrac{}{}{0pt}{}{\min}{\max}\{r, a\}$$

$\Rightarrow$ inside $\quad \Phi_M = \frac{1}{3} M r \cos\theta = \frac{1}{3} M z$

outside $\quad \Phi_M = \frac{1}{3} \frac{Ma^3}{r^2} \cos\theta$

$\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M \Rightarrow \boxed{\vec{H}_{in} = -\frac{1}{3}\vec{M}}$ inside

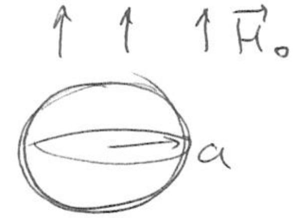
$\vec{B} = \mu_0(\vec{H} + \vec{M}) \Rightarrow \boxed{\vec{B}_{in} = \frac{2}{3}\vec{M}\mu_0}$ inside.

Outside : $\vec{H} = -\vec{\nabla}\left(\frac{1}{3}\frac{Ma^3}{r^2}\cos\theta\right)$; $\vec{B} = \mu_0\vec{H}$.

$$\Rightarrow \vec{B} = -\frac{1}{3}\mu_0 M a^3 \underbrace{\vec{\nabla}\left(\frac{\hat{z}\cdot\vec{r}}{r^3}\right)}_{=\frac{\hat{z}}{r^3} - \frac{3(\hat{z}\cdot\vec{r})\vec{r}}{r^5}} \Rightarrow \text{dipole moment}$$

makes sense $\vec{m} = \frac{4\pi a^3}{3}\vec{M}$.

<u>Example</u>: a sphere with permeable magnetic material ($\vec{B} = \mu\vec{H}$ inside, $\vec{B} = \mu_0\vec{H}$ outside) in external magnetic field $\vec{H}_0$



$\nabla^2\Phi_M = 0$ inside & outside

$\Rightarrow \Phi_{inside} = \sum_l A_l r^l P_l(\cos\theta)$, $r<a$

$\Phi_{outside} = -H_0 r P_1(\cos\theta) + \sum_{l=1}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$, $r>a$

as $H_{in,t} = H_{out,t} \Rightarrow$

$\Rightarrow (1)\ \frac{\partial\Phi_1}{\partial\theta} = \frac{\partial\Phi_2}{\partial\theta}$ at $r=a$; $B_{1n} = B_{2n} \Rightarrow \mu H_{in,n} = \mu_0 H_{out,n}$

$(2)\ \mu\left.\frac{\partial\Phi_{in}}{\partial r}\right|_{r=a} = \mu_0\left.\frac{\partial\Phi_{out}}{\partial r}\right|_{r=a}$

$P_l'(\cos\theta) = \frac{d}{d\theta}P_l(\cos\theta)$

$\Rightarrow (1)\ A_1\cdot a = -H_0 a + \frac{B_1}{a^2}$

$(2)\ \mu A_1 = -\mu_0 H_0 - \frac{2B_1\mu_0}{a^3}$

$$-H_0 + \frac{B_1}{a^3} = -\frac{\mu_0}{\mu} H_0 - \frac{2B_1}{a^3}\frac{\mu_0}{\mu}$$

$$\Rightarrow \frac{B_1}{a^3}\left(1 + 2\frac{\mu_0}{\mu}\right) = H_0\left(1 - \frac{\mu_0}{\mu}\right) \Rightarrow B_1 = H_0 a^3 \frac{\mu-\mu_0}{\mu+2\mu_0}$$

$$A_1 = -H_0 + \frac{B_1}{a^3} \Rightarrow A_1 = -\frac{3\mu_0}{\mu+2\mu_0} H_0$$

$$\Rightarrow \Phi_{inside} = -\frac{3\mu_0}{\mu+2\mu_0} H_0 r\cos\theta = -\frac{3\mu_0}{\mu+2\mu_0} H_0 z$$

$$\Phi_{outside} = -H r\cos\theta + H_0 a^3 \frac{\mu-\mu_0}{\mu+2\mu_0}\frac{1}{r^2}\cos\theta$$

& one can find $\vec{H}_{inside} = -\nabla\Phi_{inside} = \frac{3\mu_0}{\mu+2\mu_0}\vec{H}_0$

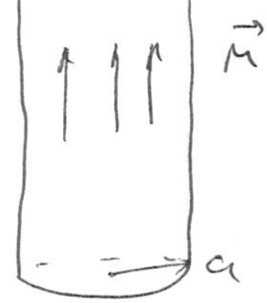
$\vec{B}_{inside} = \mu \vec{H}_{inside}$

Effective magnetization: $\vec{M} = \frac{1}{\mu_0}\vec{B}_{inside} - \vec{H}_{inside}$

$$= \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}_{inside} \Rightarrow \vec{M} = 3\frac{\mu-\mu_0}{\mu+2\mu_0}\vec{H}_0$$ cf. Jackson (5.115)

<u>Example</u>: infinite cylindrical bar magnet:

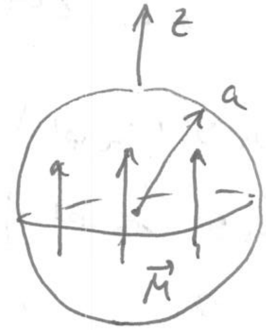
find $\vec{B}$, $\vec{H}$.



$$\Phi_M = \frac{1}{4\pi}\oint_S da' \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} = 0$$

$\Rightarrow \vec{H} = 0$, $\vec{B}_{inside} = \mu_0(\vec{H}+\vec{M}) = \mu_0\vec{M}$, $\vec{B}_{outside} = 0$.

Example: Ferromagnetic sphere, take z.



$\vec{M} = M\hat{z}$ inside the sphere

$\Rightarrow \vec{M} = M\hat{z}\,\Theta(a-r).$

use $$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\vec{\nabla}\times\vec{M} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & M_z \end{vmatrix} = \hat{x}\,\partial_y M_z - \hat{y}\,\partial_x M_z$$

$$= \hat{x}\, M\, \partial_y\, \Theta\left(a-\sqrt{x^2+y^2+z^2}\right) - \hat{y}\, M\, \partial_x\, \Theta\left(a-\sqrt{x^2+y^2+z^2}\right) =$$

$$= \hat{x}\, M\, \delta(a-r)\,\frac{-y}{r} - \hat{y}\, M\, \delta(a-r)\,\frac{-x}{r} =$$

$$= \left[-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi\right] M\,\delta(a-r) = M\sin\theta\,\delta(r-a)\cdot \underbrace{\left[-\hat{x}\sin\varphi + \hat{y}\cos\varphi\right]}_{\hat{\varphi}}$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_0^\infty dr'\,r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\varphi' \,\cdot$$

$$\cdot \underbrace{4\pi \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\, Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta,\varphi)}_{\frac{1}{|\vec{x}-\vec{x}'|}} \cdot M\,\delta(r'-a)\sin\theta'\,\hat{\varphi}'$$

now $r_{\gtrless} = \begin{matrix}\max\\ \min\end{matrix}\{r, a\}$

$$= M\mu_0 a^2 \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\varphi' \sin\theta'\,\hat{\varphi}'\, Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

$$= \mu_0 M a^2 \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\varphi' \, Y_{lm}^*(\theta',\varphi') \cdot$$

$$\underbrace{\begin{pmatrix} -\sin\theta' \sin\varphi' \\ \sin\theta' \cos\varphi' \\ 0 \end{pmatrix}}_{\begin{pmatrix} +\sqrt{\frac{8\pi}{3}}\left(Y_{11} + Y_{1,-1}\right) \cdot \frac{1}{2i} \\ -\sqrt{\frac{8\pi}{3}}\left(Y_{11} - Y_{1,-1}\right) \frac{1}{2} \\ 0 \end{pmatrix}} \Rightarrow$$

$$\Rightarrow \vec{A} = \mu_0 M a^2 \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) \begin{pmatrix} \frac{1}{2i}\sqrt{\frac{8\pi}{3}}\, \delta_{l1}\left(\delta_{m1} + \delta_{m,-1}\right) \\ -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\, \delta_{l1}\left(\delta_{m1} - \delta_{m,-1}\right) \\ 0 \end{pmatrix}$$

$$= \mu_0 M a^2 \frac{1}{3} \frac{r_<}{r_>^2} \underbrace{\begin{pmatrix} -\sin\theta \sin\varphi \\ \sin\theta \cos\varphi \\ 0 \end{pmatrix}}_{= \sin\theta\, \hat{\varphi}}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0 M a^2}{3} \frac{r_<}{r_>^2} \sin\theta\, \hat{\varphi}}$$

$\Rightarrow \vec{A}_{out} = \frac{\mu_0 M a^3}{3} \frac{\sin\theta}{r^2} \hat{\varphi}$ ~ dipole

$\vec{A}_{in} = \frac{\mu_0 M}{3} r \sin\theta\, \hat{\varphi} = \frac{\mu_0 M}{3} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$

$\Rightarrow \vec{B}_{in} = \vec{\nabla} \times \vec{A}_{in} = \frac{2}{3} \mu M \hat{z}$ ~ uniform field

$\Rightarrow$ can find $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$ ~ same as before