Midterm Review

Dielectrics: $\left(\Phi_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\vec{x}}{r^3} \Rightarrow \vec{E}_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3} \right)$

$\vec{P}$ = polarization (dipole moment per unit volume)

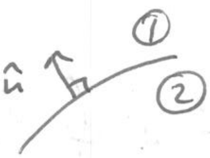
$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ displacement

$\vec{\nabla}\cdot\vec{D} = \rho_{free}$, $\vec{\nabla}\times\vec{E} = 0$ Electrostatics of dielectrics

LIH dielectrics: $\vec{D} = \varepsilon \vec{E}$.
← dielectric constant

Boundary matching: $E_{1t} = E_{2t}$, $D_{1n} - D_{2n} = \sigma_f$

$P_{1n} - P_{2n} = -\sigma_{bound}$

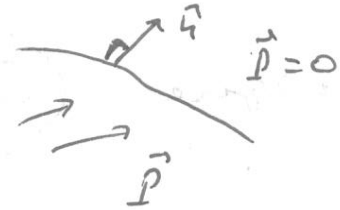


In LIH medium $\boxed{\nabla^2 \Phi = -\frac{\rho_f}{\varepsilon}}$

For medium with "frozen" polarization get

$$\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0} \int_V d^3x' \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi\varepsilon_0} \int_S da' \frac{\hat{n}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

(if the surface term is written separately)



Energy in vacuum or LIH media:

$$\boxed{W = \int d^3x \ \frac{1}{2} \vec{E}\cdot\vec{D}}$$

force: $F = -\left(\frac{\partial W}{\partial x}\right)_q$, $F = \left(\frac{\partial W}{\partial x}\right)_V$.

Magnetostatics

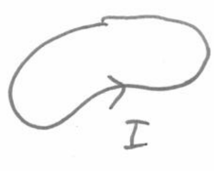
$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

charge conservation (continuity relation)

$\vec{B}$ = magnetic field

Biot & Savart Law: $\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \ \frac{\vec{J}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$

or $\vec{B} = \frac{\mu_0}{4\pi} I \oint_C \frac{d\vec{\ell} \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$



can always replace

$$I\, d\vec{\ell} \leftrightarrow \vec{J}\, d^3x$$

Ampere's Law:

$$\boxed{\begin{aligned} \vec{F} &= \int_V d^3x \ \vec{J}(\vec{x}) \times \vec{B}(x) \\ &= \oint_C d\vec{\ell} \times B(x) \cdot I \end{aligned}}$$

~ force on a current in $\vec{B}$-field

Torque $\boxed{\vec{N} = \int d^3x \ \vec{x} \times \left(\vec{J} \times \vec{B}\right)}$

Differential equations of magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0$$ (no magnetic monopoles)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

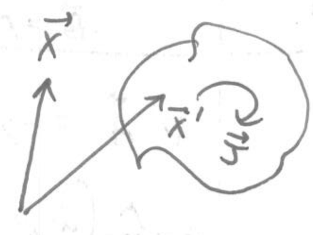
Ampere's Law (consequenc of Biot-Savart and current conservation)

Statics: $\vec{\nabla} \times \vec{B} = \mu_0 J \Rightarrow \oint_C d\vec{\ell} \cdot \vec{B} = \mu_0 I$.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ ~ vector potential

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Localized current distribution: dipole moment is leading



$$\vec{m} = \frac{1}{2} \int d^3x' \, \vec{x}' \times \vec{J}(\vec{x}')$$

dipole moment (magnetic)

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}, \qquad \vec{B} = \frac{\mu_0}{4\pi} \frac{3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}}{r^3}.$$

magnetization $\vec{M}$ = magnetic dipole moment per unit volume (density)

force $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$, $\underbrace{U = -\vec{m} \cdot \vec{B}}_{\text{energy}}$, $\underbrace{\vec{N} = \vec{m} \times \vec{B}}_{\text{torque}}$.

Magnetic field $\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}}$

$\Rightarrow$ $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ & $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

Boundary matching

$\boxed{B_{1n} = B_{2n}}$

$\boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$ ~ surface current density

$\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)$ ~ surface current density due to magnetization

LIH media: $\vec{B} = \mu \vec{H}$

method	NO FeRRomagnets $\vec{B} = \mu \vec{H}$	FeRRo magnets ($\vec{M} \neq 0$ for $\vec{H} = 0$)
$\vec{A}$	$\nabla^2 \vec{A} = -\mu \vec{J}$ $\vec{A} = \frac{\mu}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$	$\nabla^2 \vec{A} = -\mu_0 [\vec{J} + \vec{\nabla} \times \vec{M}]$ $\vec{A} = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\vec{J}(\vec{x}') + \vec{\nabla} \times \vec{M}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$ $+ \frac{\mu_0}{4\pi}\int_S da' \frac{\vec{K}(\vec{x}') - \hat{n} \times \vec{M}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$
Φ_M $\vec{H} = -\vec{\nabla}\Phi_M$ as $\vec{\nabla} \times \vec{H} = 0$ <u>for $\vec{J} = 0$</u>	$\nabla^2 \Phi_M = 0$ $\Rightarrow$ solve Laplace equation	$\nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ $\Phi_M = \frac{-1}{4\pi}\int_V d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert} + \frac{1}{4\pi}\int_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$

Energy in magnetic field $W = \frac{1}{2} \int d^3x \, \vec{H} \cdot \vec{B}$

Maxwell Equations

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \,, \qquad \vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t} \,.$$

$\Rightarrow$ in Lorenz gauge $\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \right)$ have

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Phi = -\rho/\varepsilon_0 \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{J} \qquad \text{wave equations!}$$

Green function $G_{ret} = \frac{1}{r} \delta(t - \frac{r}{c})$

$\Rightarrow$ solve

$$\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} (\vec{x}, t) = \frac{1}{4\pi} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \begin{pmatrix} \frac{\rho}{\varepsilon_0} \\ \mu_0 \vec{J} \end{pmatrix} \left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c} \right).$$

Energy & momentum:

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)$$ ~ energy density

$$\vec{S} = \vec{E}\times\vec{H}$$ Poynting vector

$$\frac{\vec{S}}{c^2} = \vec{P}_{field}$$

$$T_{ij} = \varepsilon_0\left[E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2 + c^2\left(B_i B_j - \frac{1}{2}\delta_{ij}\vec{B}^2\right)\right]$$

Maxwell stress tensor

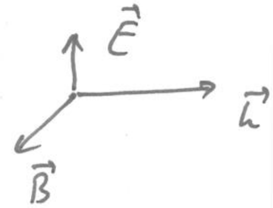
$$\frac{\partial u_f}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$ energy conservation

$$\frac{\partial}{\partial t}\left(p_{field} + p_{mech}\right)_i = \nabla_j T_{ij}$$ momentum conservation

Plane waves: $$\left[\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

$$\vec{E} = \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$ ~ monochromatic plane wave

$$\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$$



$$\langle u\rangle = \frac{1}{2}\varepsilon E_0^2\,, \quad \langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0^2\,\hat{k}$$

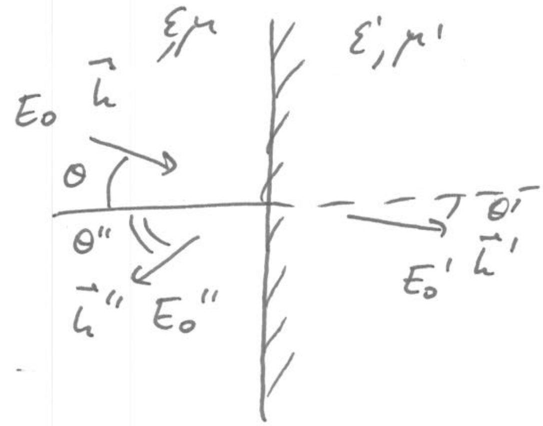
$$\vec{E} = (\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

linear polarizations

$$= (\hat{\varepsilon}_+ E_+ + \hat{\varepsilon}_- E_-)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

$$\hat{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2) \quad \sim \text{circular polarizations}$$

Reflection & refraction:



boundary matching:

B_n, D_n, E_t, H_t = continuous

$\omega = \omega' = \omega''$, $\quad k = k'' = \sqrt{\mu\varepsilon}\,\omega$

$k' = \sqrt{\mu'\varepsilon'}\,\omega$.

$\theta = \theta''$ $\qquad n \sin\theta = n' \sin\theta'$ $\quad$ Snell's Law

transmission coefficient

$$T = \frac{|\vec{S}'|}{|\vec{S}|} = \sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}} \left(\frac{E_0'}{E_0}\right)^2$$

reflection coefficient

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \left(\frac{E_0''}{E_0}\right)^2.$$