Last time

Lorentz Transformations

Def

4-dim coordinates:

$$\boxed{x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z}$$

Def $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Boost along the x-axis is

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

or

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$\mu, \nu = 0, 1, 2, 3$

Sum over repeated index is implied:

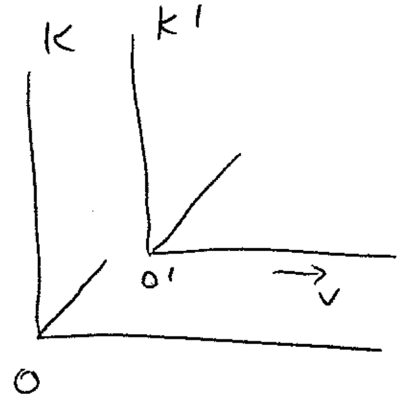
$$\Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_0 x^0 + \Lambda^\mu{}_1 x^1 + \Lambda^\mu{}_2 x^2 + \Lambda^\mu{}_3 x^3.$$

The <u>inverse</u> Lorentz transform:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

Invariant <u>interval</u>: flash a light at the origin(s) at $t = t' = 0$

$\Rightarrow$ light reaches point (x, y, z) after time t such that



$$c^2 t^2 - x^2 - y^2 - z^2 = 0$$

In moving frame (K') have

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = 0.$$

$\Rightarrow$ as $y' = y$, $z' = z$ $\Rightarrow$ $c^2 t^2 - x^2 = c^2 t'^2 - x'^2$

$\Rightarrow$ can explicitly check that it works:

$$c^2 t'^2 - x'^2 = \frac{1}{1 - \frac{v^2}{c^2}} \left[\left(ct - \frac{v}{c} x \right)^2 - (x - vt)^2 \right] =$$

$$= \frac{1}{1 - \frac{v^2}{c^2}} \left[c^2 t^2 \left(1 - \frac{v^2}{c^2} \right) - x^2 \left(1 - \frac{v^2}{c^2} \right) \right] = c^2 t^2 - x^2$$

as expected.

Quantity $S_{12}^2 = c^2 (t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$

is called the square of the *interval* between 2 events at times t_1 & t_2 & locations $\vec{x}_1$ & $\vec{x}_2$. S_{12}^2 is Lorentz-invariant.

(i) $S_{12}^2 > 0 \Rightarrow$ timelike separation $\Rightarrow$ there exists a frame where $\vec{x}_1' = \vec{x}_2' \Rightarrow S_{12}^2 = c^2(t_1'^2 - t_2'^2)$

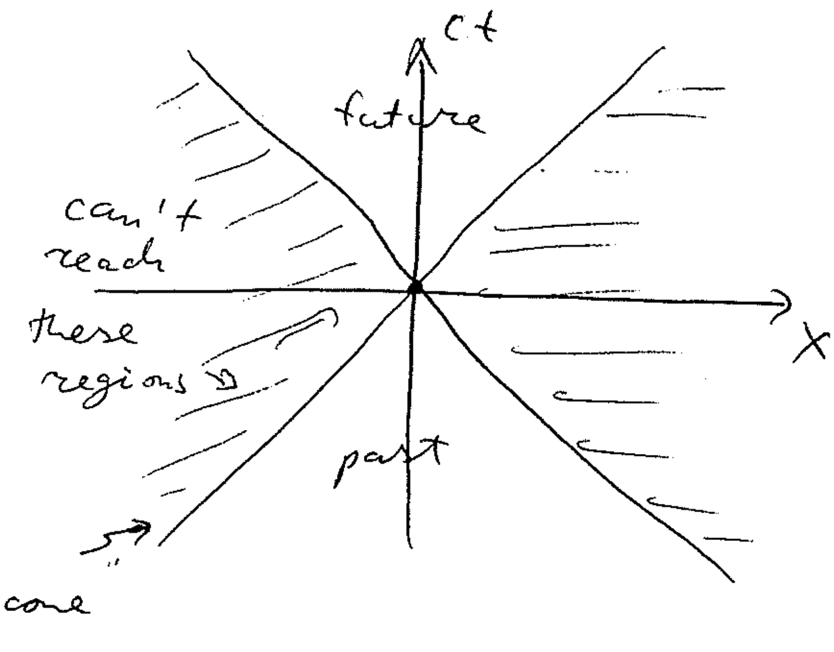
$\Rightarrow$ the events take place at the same space point, but at diff. times

(ii) $S_{12}^2 < 0 \Rightarrow$ spacelike separation $\Rightarrow$

there exists a frame where $t_1'' = t_2'' \Rightarrow$

$\Rightarrow S_{12}^2 = -(\vec{x}_1'' - \vec{x}_2'')^2 \Rightarrow$ events take place at the same time but at different locations

(iii) $S_{12}^2 = 0 \Rightarrow$ lightlike separation



Proper time & Time Dilation.

Definition Proper time is the time in the rest frame of an object: $\boxed{d\tau \equiv \frac{ds}{c}}$.

Example: imagine a frame in which a particle moves with velocity $\vec{u}(t) \Rightarrow d\vec{x} = \vec{u}(t)\,dt$

$$\Rightarrow ds^2 = c^2 dt^2 - (d\vec{x})^2 = c^2 dt^2 - \vec{u}^2 dt^2 =$$

$$= c^2 dt^2 \left(1 - \beta^2(t)\right).$$

In the rest frame of the particle

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\beta^2(t)} \Rightarrow \boxed{\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}}.$$

alternatively

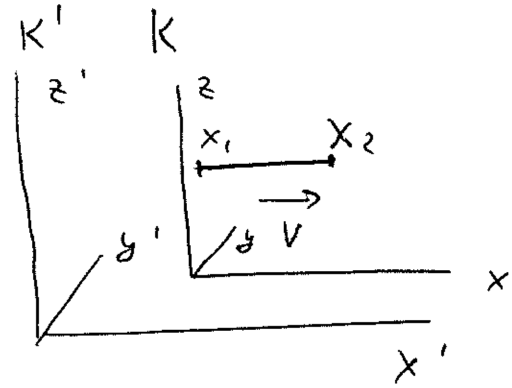
$$\boxed{t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}}} \Rightarrow \Delta t \geqslant \Delta\tau \text{ - time dilation.}$$

(E.g.: a photon has $\beta = 1 \Rightarrow \Delta\tau = 0 \Rightarrow$

$\Rightarrow$ the whole lifetime of the Universe is instanteneous for a photon!)

Lorentz Contraction.

Imagine a bar (rod) moving at a constant velocity (see figure).



Proper length is defined as its length in the rest frame of the bar (frame K).

$$\Rightarrow x_1 = \frac{x_1' - vt'}{\sqrt{1-\frac{v^2}{c^2}}} \,, \quad x_2 = \frac{x_2' - vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta x = x_2 - x_1 = \frac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\Rightarrow$ if l_0 is proper length, $l_0 = \Delta x$

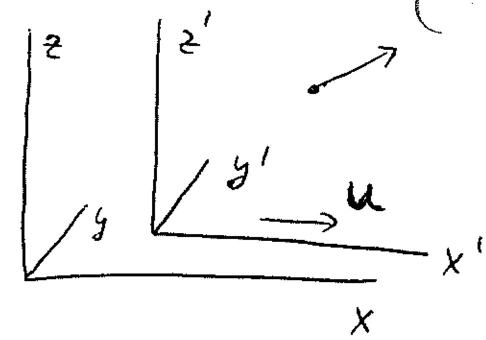
$$\Rightarrow l = \Delta x' \Rightarrow l = l_0 \sqrt{1-\frac{v^2}{c^2}}$$

$\Rightarrow l \leq l_0$ Lorentz contraction.

~ objects appear shorter in other frames. (as compared to the rest frame)

Velocity Transformations.

$$\begin{cases} x = \gamma\,(x' + u t') \\ y = y' \\ z = z' \\ ct = (t'c + \beta x')\,\gamma \end{cases}$$



$$\Rightarrow V_x = \frac{dx}{dt} = \frac{dx' + u\,dt'}{dt' + \frac{\beta}{c}\,dx'} = \frac{V_x' + u}{1 + \frac{u V_x'}{c^2}} = V_x$$

where $V_x' = \dfrac{dx'}{dt'}$.

$$V_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\beta}{c}\,dx'\right)} = \frac{V_y'}{\gamma\left(1 + \frac{\beta}{c} V_x'\right)} = \frac{V_y' \sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{u V_x'}{c^2}} = V_y$$

Similarly $$V_z = \frac{V_z' \sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{u V_x'}{c^2}}$$

($c \to \infty$ get Galilean $V_x = V_x' + u$, $V_y' = V_y$, $V_z' = V_z$)

Imagine the case when $V_x = V\cos\theta$, $V_y = V\sin\theta$, $V_z = 0$

$$\Rightarrow V_z' = 0\,, \quad \tan\theta = \frac{V_y}{V_x} = \frac{V_y'}{\gamma\,(V_x' + u)} \Rightarrow$$

$\Rightarrow$ write $V_x' = V'\cos\theta'$

$V_y' = V'\sin\theta'$

$$\Rightarrow \boxed{\tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + u)}}$$

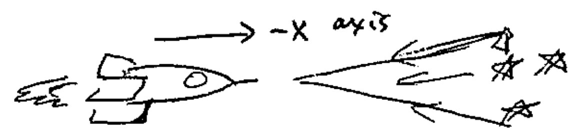
<u>Light abberation</u>: if the particle is a photon $\Rightarrow v = v' = c \Rightarrow \tan\theta = \dfrac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$, $\beta = \frac{u}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$\Rightarrow V_x = \frac{V_x' + u}{1 + \frac{uV_x'}{c^2}} \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

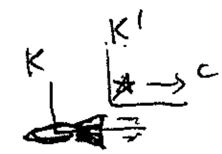
$$V_y = \frac{V_y'}{\gamma\left(1 + \frac{uV_x'}{c^2}\right)} \Rightarrow \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}$$

$\Rightarrow$ if $u \to c \Rightarrow \underset{(\text{as } \gamma \to \infty)}{\sin\theta \to 0}$, $\cos\theta \to 1 \Rightarrow$

$\Rightarrow \theta \to 0 \Rightarrow$ For UR spaceship all stars appear in the small cone ahead, at $\theta = 0$.



K ~ space ship rest frame, K′ ~ Universe, $\theta = 0$ means the light comes along +x axis



~~Four-vectors.~~

~~$x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$~~

~~(x_0, x_1, x_2, x_3) is a 4-vector.~~