**Last time** The Lagrangian and the Action (cont'd)

Constructed the Lagrangian for a relativistic particle in electromagnetic field:

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

Wrote down equations of motion for this Lagrangian:

$$\boxed{\frac{dp^i}{dt} = \frac{e}{c}\left[c\left(\frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x^0}\right) - v^j\left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right)\right]}$$

Def. Electric field $\boxed{\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}$ $\Rightarrow E^i = \partial^i A^0 - \partial^0 A^i$

Def. Magnetic field $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ $\Rightarrow \partial^j A^i - \partial^i A^j = \varepsilon^{ijk} B^k$

$\Rightarrow$ using these definitions we wrote the EOM as

$$\boxed{\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}}$$ Lorentz force!

$\Rightarrow$ We obtained the Lorentz force from the Lorentz invariance ⊕ homogeneity & isotropy of space-time ⊕ existence of a 4-vector field $A^\mu(x)$.

**Def.** Field-strength tensor:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

$F_{\mu\nu} = -F_{\nu\mu}$ anti-symmetric

Components are: $F^{0i} = \partial^0 A^i - \partial^i A^0 =$

$$= \frac{\partial A^i}{\partial x^0} - \frac{\partial A^0}{\partial x_i} = -E^i \quad \Rightarrow \quad F^{0i} = -E^i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j} = -\varepsilon^{ijk} B^k$$

$$\Rightarrow \quad F^{ij} = -\varepsilon^{ijk} B^k$$

$\Rightarrow$ components of the field-strength tensor are the electric and magnetic fields!

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

where $E_x = E^1, E_y = E^2, E_z = E^3,$ $B_x = B^1, B_y = B^2, B_z = B^3.$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

will revisit later

$\Rightarrow$ $\vec{E}$ & $\vec{B}$ fields transform as components of a rank-2 tensor under Lorentz transformations!

Using $F^{\mu\nu}$ we can write the Lorentz force

as

$$\frac{dp^i}{dt} = \frac{e}{c}\left[-c\,F^{0i} - v^j F^{ij}\right] = \frac{e}{c}\left[cF^{i0} - v^j F^{ij}\right]$$

$$= \frac{e}{c}\,\frac{1}{\gamma}\,u_\mu F^{i\mu} \quad \Rightarrow \text{ as } dt = \gamma\, d\tau$$

$$\Rightarrow \frac{dp^i}{d\tau} = \frac{e}{c}\,u_\mu\,F^{i\mu}$$

what about $\frac{dp^0}{d\tau}$? as $p^\mu = m u^\mu$ and $u_\mu u^\mu = c^2$

$$\Rightarrow u_\mu \frac{dp^\mu}{d\tau} = 0 \quad (\text{why?}) \quad \Rightarrow c\,\frac{dp^0}{d\tau} - v^i \frac{dp^i}{d\tau} = 0$$

$$\Rightarrow \frac{dp^0}{d\tau} = \frac{1}{c}\,v^i\,\frac{dp^i}{d\tau} = \frac{1}{c}\,\frac{e}{c}\,\gamma v^i\left[c\,F^{i0} - \underbrace{\cancel{v^j F^{ij}}}_{0}\right]$$

as $v^i v^j F^{ij} = 0$

$$= \frac{e}{c}\,\gamma\, v^i F^{i0} = \frac{e}{c}\,u_\mu F^{0\mu} \leftarrow (\text{as } F^{00} = 0)$$

$$\Rightarrow \frac{dp^0}{d\tau} = \frac{e}{c}\,u_\mu\,F^{0\mu}$$

$\Rightarrow$ can combine the two into

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{e}{c}\,u_\nu F^{\mu\nu}}$$

Lorentz-covariant form of the Lorentz force

In the non-relativistic notation we write for $p^0$:

$$\frac{dp^0}{d\tau} = \frac{1}{c} v^i \frac{dp^i}{d\tau} \Rightarrow \frac{dp^0}{dt} = \frac{1}{c} v^i \frac{dp^i}{dt} \Rightarrow \text{as } p^0 = \frac{E}{c}$$ ← energy of the particle

$$\Rightarrow \frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt} = \vec{v} \cdot \left[ e\vec{E} + \frac{e}{c} \vec{v} \times \vec{B} \right] = e\,\vec{v} \cdot \vec{E}$$

$$\Rightarrow \boxed{\frac{dE}{dt} = e\,\vec{v} \cdot \vec{E}}$$ ⇒ only electric field does the work on the particle.

(Note that $\frac{dE}{dt} = \vec{F} \cdot \vec{v}$ as we saw before ⇒

⇒ as $\vec{F} = \frac{d\vec{p}}{dt}$ (the force) ⇒ above is consistent with the formulas for energy change studied earlier.)

We can also construct the Hamiltonian for the particle in the E&M fields. Since the canonical momentum is $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$

$$\Rightarrow H = \vec{P} \cdot \vec{v} - L \Big|_{\text{with } \vec{v} \text{ replaced by } \vec{P}} \Rightarrow$$

as $\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}\vec{A} \Rightarrow \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 = \frac{m^2 v^2}{1 - \frac{v^2}{c^2}}$

$$\Rightarrow \quad v^2 \left[ m^2 + \frac{1}{c^2} \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2 \right] = \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2$$

$$\Rightarrow \quad \vec{v} = \frac{c \left( \vec{P} - \frac{e}{c} \vec{A} \right)}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}}$$

(also need that $\vec{v} \parallel \vec{P} - \frac{e}{c} \vec{A}$ )

$$\Rightarrow H = \vec{P} \cdot \frac{c \left( \vec{P} - \frac{e}{c} \vec{A} \right)}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}} + m c^2 \sqrt{1 - \frac{v^2}{c^2}} + e \Phi - \frac{e}{c} \vec{v} \cdot \vec{A}$$

$$= \vec{P} \cdot \frac{c \left( \vec{P} - \frac{e}{c} \vec{A} \right)}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}} + \frac{m^2 c^3}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}} + e \Phi$$

$$- \frac{e}{c} \vec{A} \cdot \frac{c \left( \vec{P} - \frac{e}{c} \vec{A} \right)}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}} = c \frac{\left( \vec{P} - \frac{e}{c} \vec{A} \right)^2 + m^2 c^2}{\sqrt{m^2 c^2 + \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}} + e \Phi$$

$\Rightarrow$ the Hamiltonian is

$$\boxed{H = \underbrace{\sqrt{m^2 c^4 + c^2 \left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}}_{\sqrt{m^2 c^4 + p^2 c^2}} + e \Phi} \quad \Rightarrow \text{ total energy}$$

of the particle!

Non-relativistic limit can be obtained by expansion:

$$H \simeq m c^2 + \frac{\left( \vec{P} - \frac{e}{c} \vec{A} \right)^2}{2m} + e \Phi$$

~ may be familiar from QM

in QM quantize the system by replacing $\vec{P} \to -i \hbar \vec{\nabla}$.