Start out with microscopic equations for magnetostatics ($\vec{b}$ ~ magnetic field, $\vec{j}$ ~ current)

$$\boxed{\vec{\nabla}\cdot\vec{b} = 0} \quad \text{and} \quad \boxed{\vec{\nabla}\times\vec{b} = \mu_0 \vec{j}}$$

Introduce averaging in space:

$$\langle \vec{j}(\vec{x}) \rangle = \int \frac{d^3x'}{V}\, \vec{j}(\vec{x}-\vec{x}')\, f(\vec{x}')$$

where $f(\vec{x}')$ is some profile function: 

e.g. $f(\vec{x}) = \begin{cases} 1, & |\vec{x}| < R \\ 0, & \text{otherwise} \end{cases}$; $V = \frac{4}{3}\pi R^3$.

⇒ averaging represents "smearing" over some region of radius R!

Now, $$\vec{j}(\vec{x}) = \sum_n \sum_j q_j \left(\vec{v}_n + \vec{v}_{nj}\right) \delta(\vec{x} - \vec{x}_n - \vec{x}_{nj})$$

$\sum_n$ ↗ all molecules; $\sum_j$ ↖ all electrons/protons in molecules

$\vec{x}_n$ ~ position of nth molecule

$\vec{v}_n$ ~ velocity — " —

$\vec{x}_{nj}$ ~ position of charge i in the nth molecule with respect to its center

$\vec{v}_{nj}$ ~ velocity — " —

$$\langle \vec{j}(\vec{x}) \rangle = \left\langle \sum_{n,j} q_j \, (\vec{v}_n + \vec{v}_{nj}) \, \delta(\vec{x} - \vec{x}_n - \vec{x}_{nj}) \right\rangle =$$

$$\simeq \left\langle \sum_{n,j} q_j \, (\vec{v}_n + \vec{v}_{nj}) \left[\delta(\vec{x} - \vec{x}_n) - \vec{x}_{nj} \cdot \vec{\nabla} \, \delta(\vec{x} - \vec{x}_n) + \ldots \right] \right\rangle$$

Def. Macroscopic current

$$\vec{J}(\vec{x}) \equiv \left\langle \sum_{n,j} q_j \, (\vec{v}_n + \vec{v}_{nj}) \, \delta(\vec{x} - \vec{x}_n) \right\rangle$$

Def. Macroscopic magnetization

$$\vec{M}(\vec{x}) \equiv \left\langle \sum_n \vec{m}_n \, \delta(\vec{x} - \vec{x}_n) \right\rangle =$$

$$= \left\langle \sum_{n,j} \frac{1}{2} q_j \, (\vec{x}_{nj} \times \vec{v}_{nj}) \, \delta(\vec{x} - \vec{x}_n) \right\rangle$$

Remember: $\vec{\mu}(\vec{x}) = \frac{1}{2} \, \vec{x} \times \vec{j}(\vec{x})$ microscopically

$\Rightarrow$ macroscopically $\vec{M}(\vec{x}) = \langle \vec{\mu}(\vec{x}) \rangle$

$$= \frac{1}{2} \langle \vec{x} \times \vec{j}(\vec{x}) \rangle = \frac{1}{2} \int \frac{d^3x'}{V} \, (\vec{x} - \vec{x}') \times \vec{j}(\vec{x} - \vec{x}') \, f(\vec{x}')$$

$$= \int \frac{d^3x'}{V} \, \frac{1}{2} \sum_{n,j} q_j (\vec{x} - \vec{x}' \underset{-\vec{x}_n}{\wedge}) \times (\vec{v}_n + \vec{v}_{nj}) \, \delta(\vec{x} - \vec{x}' - \vec{x}_n - \vec{x}_{nj}) \, f(\vec{x}')$$

$-\vec{x}_n$ ~ shift the origin

$|\vec{v}_n| \ll |\vec{v}_{nj}| \Rightarrow$ neglect

$$\Rightarrow \vec{M}(\vec{x}) = \frac{1}{2V} \sum_{n,j} q_j \left(\vec{x}_{nj} \times \vec{v}_{nj}\right) f(\vec{x} - \vec{x}_n - \vec{x}_{nj})$$

$$\approx \frac{1}{2} \frac{1}{V} \sum_{n,j} q_j \left(\vec{x}_{nj} \times \vec{v}_{nj}\right) f(\vec{x} - \vec{x}_n) + o(x_{nj})$$

$$\Rightarrow \vec{M}(\vec{x}) = \left\langle \frac{1}{2} \sum_{n,j} q_j \left(\vec{x}_{nj} \times \vec{v}_{nj}\right) \delta(\vec{x} - \vec{x}_n) \right\rangle$$

as defined $\Rightarrow$ it's consistent!

With the above definitions:

$$\boxed{\langle \vec{j}(\vec{x}) \rangle = \vec{J}(\vec{x}) + \vec{\nabla} \times \vec{M}(\vec{x})}$$

Check:

$$\vec{\nabla} \times \vec{M} = \left\langle \frac{1}{2} \sum_{n,j} q_j \, \vec{\nabla} \times \left[\left(\vec{x}_{nj} \times \vec{v}_{nj}\right) \delta(\vec{x} - \vec{x}_n) \right] \right\rangle$$

$$= \left\langle \sum_{n,j} \frac{1}{2} q_j \left[\vec{x}_{nj} \left(\vec{v}_{nj} \cdot \vec{\nabla}\right) \delta(\vec{x} - \vec{x}_n) - \vec{v}_{nj} \left(\vec{x}_{nj} \cdot \vec{\nabla}\right) \cdot \delta(\vec{x} - \vec{x}_n) \right] \right\rangle = \left(\text{similar to } \int d^3x \left[x_i J_j + x_j J_i \right] = 0\right)$$

$$= - \left\langle \sum_{n,j} q_j \, \vec{v}_{nj} \left(\vec{x}_{nj} \cdot \vec{\nabla}\right) \delta(\vec{x} - \vec{x}_n) \right\rangle \approx \left(\text{as } v_n \ll v_{nj}\right)$$

$$\approx - \left\langle \sum_{n,j} q_j \left(\vec{v}_n + \vec{v}_{nj}\right) \left(\vec{x}_{nj} \cdot \vec{\nabla}\right) \delta(\vec{x} - \vec{x}_n) \right\rangle .$$

Finally, defining macroscopic magnetic field

$$\vec{B}(\vec{x}) \equiv \langle \vec{b}(\vec{x}) \rangle$$

get: $\vec{\nabla}\cdot\vec{b} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

$$\vec{\nabla}\times\vec{b} = \mu_0 \vec{j} \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \langle \vec{j} \rangle = \mu_0 [\vec{J} + \vec{\nabla}\times\vec{M}]$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0 [\vec{J} + \vec{\nabla}\times\vec{M}]}$$

In principle one can define $\vec{M}$ by $\vec{J}_M = \vec{\nabla}\times\vec{M}$ with $\vec{J}_M$ effective current due to bound currents. Then $\vec{M} \to \vec{M} + \vec{\nabla}\psi$ leaves $\vec{J}_M$ unchanged $\Rightarrow$ ambiguity in definition. $\vec{M} = \frac{1}{2}\vec{x}\times\vec{J}_M$ and $\vec{J} = \vec{\nabla}\times\vec{M}$ are related by "gauge" transformation:

$$m_i = \frac{1}{2}\int d^3x\, (\vec{x}\times\vec{J}_M)_i = \frac{1}{2}\int d^3x\, [\vec{x}\times(\vec{\nabla}\times\vec{M})]_i =$$

$$= \frac{1}{2}\int d^3x\, [x_j \nabla_i M_j - x_j \nabla_j M_i] \overset{(parts)}{=}$$

$$= \frac{1}{2}\int d^3x\, [-M_i + 3M_i] = \int d^3x\, M_i$$

$$\Rightarrow \int d^3x\, \frac{1}{2}\vec{x}\times\vec{J}_M = \int d^3x\, \vec{M}$$