

Multipole Expansion.

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$



localized charges

Use

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_c^l}{r_l^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi).$$

to obtain

$$\phi(\vec{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \left[\int d^3x' Y_{lm}^*(\theta', \varphi') r'^l \rho(\vec{x}') \right].$$

$$Y_{lm}(\theta, \varphi) \frac{1}{r^{l+1}}, \text{ where we've used the}$$

fact that far from the charges $r_l = r, r_c = r'$ Definition Defining multipole moments

$$q_{lm} = \int d^3x' Y_{lm}^*(\theta', \varphi') r'^l \rho(\vec{x}')$$

we obtain

$$\phi(\vec{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)$$

multipole
expansion

useful at large r , where the expansion parameter
 $\sim 1/r$ is small

Some low order multipole moments:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \Rightarrow q_{00} = \frac{1}{\sqrt{4\pi}} \int d^3x' \rho(\vec{x}') = \frac{q}{\sqrt{4\pi}},$$

where q is the total charge in the system.

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta (\cos\varphi + i\sin\varphi) = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$$

$$Y_{10} = +\sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

as $Y_{lm}^* \leftarrow c.c.$

$$\begin{aligned} \Rightarrow q_{11} &= -\sqrt{\frac{3}{8\pi}} \int d^3x' \rho(\vec{x}') (x' - iy') = \\ &= -\sqrt{\frac{3}{8\pi}} (p_x - ip_y) \end{aligned}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int d^3x' \rho(\vec{x}') z' = \sqrt{\frac{3}{4\pi}} p_z$$

where we defined electric dipole moment (EDM)

$$\boxed{\vec{p} \equiv \int d^3x' \rho(\vec{x}') \vec{x}'}$$

$$\begin{aligned} Y_{22} &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\varphi} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta (\cos^2\varphi - \\ &- \sin^2\varphi + 2i\sin\varphi \cos\varphi) = \frac{1}{4r^2} \sqrt{\frac{15}{2\pi}} (x^2 - y^2 + 2ixy) \end{aligned}$$

$$\Rightarrow q_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') (x' - iy')^2 =$$

$$= \frac{1}{12} \sqrt{\frac{15}{25}} (Q_{11} - 2iQ_{12} - Q_{22})$$

where we've defined (traceless) quadrupole moment tensor

$$Q_{ij} = \int d^3x' \rho(\vec{x}') [3x'_i x'_j - r'^2 \delta_{ij}]$$

By analogy, $Y_{21} = -\sqrt{\frac{15}{85}} \sin\theta \cos\theta e^{i\varphi} =$

$$= -\sqrt{\frac{15}{85}} \frac{z}{r^2} (x + iy) \Rightarrow g_{21} = -\sqrt{\frac{15}{85}} \int d^3x' \rho(\vec{x}') \cdot$$

$$z' (x' - iy') = -\frac{1}{3} \sqrt{\frac{15}{85}} (Q_{13} - iQ_{23})$$

One can also show that $g_{20} = \frac{1}{2} \sqrt{\frac{5}{45}} Q_{33}$.

As $Y_{\ell,-m}(\theta, \varphi) = (-1)^m Y_{\ell m}^*(\theta, \varphi) \Rightarrow g_{\ell,-m} = (-1)^m g_{\ell m}^*$

can use to obtain other $g_{\ell,-m}$'s.

Using the found multipole moments we get

$$\phi(\vec{x}) = \frac{1}{\epsilon_0} \frac{q_{00}}{r} Y_{00} + \frac{1}{3\epsilon_0} \frac{1}{r^2} (q_{11} Y_{11} + q_{10} Y_{10} + q_{1,-1} Y_{1,-1}) +$$

$$+ \dots = \frac{1}{4\pi\epsilon_0} \frac{q}{r} + \frac{1}{3\epsilon_0} \frac{3}{8\pi} \frac{1}{r^2} \left(+ (p_x - ip_y) \frac{x+iy}{r} + \right.$$

$$\left. + (p_x + ip_y) \frac{x-iy}{r} + 2p_z \frac{z}{r} \right) + \dots \Rightarrow$$

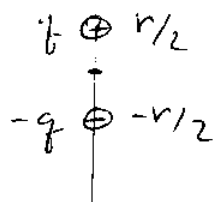
$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r} + \frac{\vec{p} \cdot \vec{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]$$

can also be derived.

Examples:

dipole: $q=0$

$$\vec{p} = q \frac{\vec{r}}{2} - (-q) \left(-\frac{\vec{r}}{2}\right) = q \vec{r} \text{ v.o.k.}$$

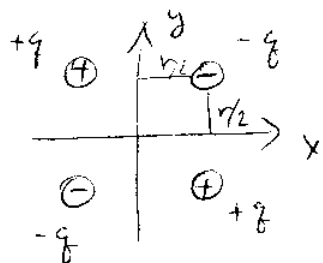


$$Q_{zz} = 3\left(\frac{r}{2}\right)^2 q - 3\left(\frac{r}{2}\right)^2 q - \left(\frac{r}{2}\right)^2 q + \left(\frac{r}{2}\right)^2 q = 0$$

$$\text{all } Q_{ij} = 0, \dots$$

quadrupole:

$$q=0$$



$$\Rightarrow \vec{p} = 0$$

$$Q_{xx} = -q \cdot \left(3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right) + q \cdot \left(3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right) = 0$$

$$Q_{xy} = -q \cdot 3\left(\frac{r}{2}\right)^2 \cdot 2 + q \cdot 3\left(\frac{r}{2}\right)\left(-\frac{r}{2}\right) \cdot 2 = -3qr^2 \neq 0$$

$$Q_{yy} = 0.$$

non-zero quadrupole moment.

If we know Φ , we know $\vec{E} = -\vec{\nabla} \Phi \Rightarrow$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r^3} \vec{r} + \frac{3\hat{n}(\vec{p} \cdot \hat{n}) - \vec{p}}{r^3} + \dots \right]$$

dipole's electric field

$$\hat{n} = \frac{\vec{r}}{r}$$