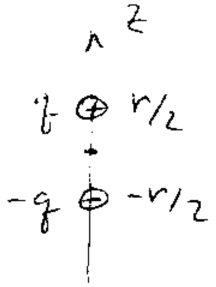
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots\right]$$

can also be derived.

Examples:



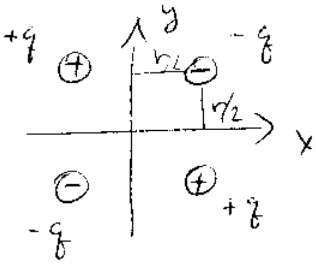
dipole: $q = 0$

$$\vec{p} = q\,\frac{\vec{r}}{2} - (-q)\left(-\frac{\vec{r}}{2}\right) = q\vec{r} \quad \text{vou.}$$

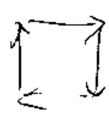
$$Q_{zz} = 3\left(\frac{r}{2}\right)^2 q - 3\left(\frac{r}{2}\right)^2 q - \left(\frac{r}{2}\right)^2 q + \left(\frac{r}{2}\right)^2 q = 0$$

all $Q_{ij} = 0, \dots$

quadrupole:



$=$



$q = 0$

$\Rightarrow \vec{p} = 0$

$$Q_{xx} = -q\cdot\left(3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right)2 + q\left(+3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right) = 0$$

$$Q_{xy} = -q\cdot 3\left(\frac{r}{2}\right)^2\cdot 2 + q\cdot 3\left(\frac{r}{2}\right)\left(-\frac{r}{2}\right)\cdot 2 = -3qr^2 \neq 0.$$

$Q_{yy} = 0.$

non-zero quadrupole moment.

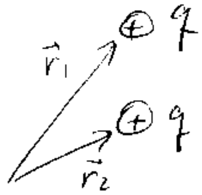
If we know Φ, we know $\vec{E} = -\vec{\nabla}\Phi \Rightarrow$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^3}\vec{r} + \underbrace{\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{r^3}}_{\text{dipole's electric field}} + \dots\right]$$

$\hat{n} = \frac{\vec{r}}{r}.$

All, but the <u>lowest non-vanishing</u> multipole moments depend on the choice of origin!

e.g.



$$q_{00} = \frac{2q}{\sqrt{4\pi}} \neq 0$$

$$q_{1m} \to \vec p = q\vec r_1 + q\vec r_2 \quad \text{depends on origin}$$

$\vdots$

etc.

However, we've been a bit sloppy: for a dipole

$$\Phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec p\cdot\vec r}{r^3} = \frac{-1}{4\pi\varepsilon_0}\vec p\cdot\vec\nabla\frac{1}{r}$$

Start by regulating $\quad \frac{1}{r} = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{x^2+y^2+z^2+\varepsilon^2}}$

$$E_i = +\frac{1}{4\pi\varepsilon_0} p_j \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{\sqrt{x^2+y^2+z^2+\varepsilon^2}} = \frac{1}{4\pi\varepsilon_0} p_j \cdot$$

$$\cdot\left[\frac{3x_i x_j}{(r^2+\varepsilon^2)^{5/2}} - \frac{\delta_{ij}}{(r^2+\varepsilon^2)^{3/2}}\right] = \frac{1}{4\pi\varepsilon_0} p_j \frac{3x_i x_j - \delta_{ij}(r^2+\varepsilon^2)}{(r^2+\varepsilon^2)^{5/2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3x_i\,\vec p\cdot\vec r - p_i r^2}{r^5} - \frac{1}{4\pi\varepsilon_0} p_i \underbrace{\frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}}_{\delta_\varepsilon(r)}$$

(i) $r \neq 0,\ \varepsilon\to 0 \Rightarrow \delta_\varepsilon \to 0$

$\quad r = 0,\ \varepsilon \to 0 \Rightarrow \delta_\varepsilon(0) = \infty$

(ii) $$\int d^3r\, \frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = 4\pi\int_0^\infty dr\, r^2 \frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = \Big|\, \tilde r = \frac{r}{\varepsilon}$$

$$= 4\pi \int_0^\infty d\tilde{r} \, \frac{\tilde{r}^2}{(1+\tilde{r}^2)^{5/2}} = \frac{4\pi}{3} \Rightarrow \delta_\varepsilon(r) \xrightarrow[\varepsilon \to 0]{} \frac{4\pi}{3} \delta^3(\vec{r})$$

$\Rightarrow$ field of a dipole is, in fact:

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \Big[\underbrace{\frac{3\vec{r}\,(\vec{p}\cdot\vec{r}) - \vec{p}\,r^2}{r^5}}_{\text{part known from undergraduate E\&M}} - \underbrace{\frac{4\pi}{3} \vec{p}\, \delta^3(\vec{r})}_{\text{new piece. makes } \int d^3x\, \vec{E} \text{ finite.}} \Big]}$$

Electrostatic energy & multipole expansion

put our system of charges in external field $\Phi(\vec{x})$: potential energy will be

$$W = \int d^3x \, \rho(\vec{x}) \, \Phi(\vec{x})$$

Suppose $\Phi(\vec{x})$ is slowly varying $\Rightarrow$ expand

$$\Phi(\vec{x}) \simeq \Phi(0) + \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2} \sum_{i,j} x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(0) + \dots$$

$\Rightarrow$ as $\vec{E} = -\vec{\nabla}\Phi$ and as $\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi = \frac{1}{\varepsilon_0}\rho_{ext} = 0$

in absense of external charges $\Rightarrow$

$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E} - \frac{1}{6} \sum_{i,j} \left(3x_i x_j - r^2 \delta_{ij}\right) \frac{\partial E_j}{\partial x_i}(0) + \dots$$

Therefore,

$$\boxed{W = q\,\Phi(0) - \vec{p}\cdot\vec{E} - \frac{1}{6}\sum_{i,j} Q_{ij}\,\frac{\partial E_j}{\partial x_i}(0) + \dots}$$

Dielectrics.

Suppose we have two types of charges:

"free charges" and "bound charges".

The potential is then the sum of potentials of free and bound charges: $\Phi = \Phi_{free} + \Phi_{bound}$

$$\Phi_{free}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho_f(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Bound charges give charge-neutral media (e^- & p in the atoms & molecules). The dominant multipole is dipole. (It's easy to polarize a molecule.) Potential of a point dipole $\vec{p}$ (at $\vec{x}'$) is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Def. Defining <u>polarization</u> $\vec{P}(\vec{x})$ as dipole moment per unit volume, we write

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \; \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

where V is the region containing polarization $\vec{P}$.

As $\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} \Rightarrow$

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \; \vec{P}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|} = (\text{parts})$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d^3x' \; \frac{1}{|\vec{x}-\vec{x}'|} \; \vec{\nabla}'\cdot\vec{P}(\vec{x}')$$

Finally, $\Phi(\vec{x}) = \Phi_{free}(\vec{x}) + \Phi_{bound}(\vec{x}) =$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3x' \; \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho_f(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')\right].$$

$\Rightarrow$ There seems to be two components to <u>total</u> charge density: $\rho_{tot} = \rho_f - \vec{\nabla}\cdot\vec{P}$

Now, $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ true in dielectrics

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi = \frac{1}{\varepsilon_0}\left[\rho_f(\vec{x}) - \vec{\nabla}\cdot\vec{P}(\vec{x})\right]$$

as $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

Def. Define electric displacement $\vec{D}$

as $$\boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$$

Since $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\left[\rho_f - \vec{\nabla}\cdot\vec{P}\right]$

$$\Rightarrow \vec{\nabla}\cdot\left[\varepsilon_0\vec{E} + \vec{P}\right] = \rho_f \Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

$\vec{D}$ appears to have the meaning of electric field due to free charges.

Linear Isotropic Homogeneous medium (LIH):

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

linear ~ as $\vec{P} \propto \vec{E}$

homogeneous: χ is a constant, and not $\chi(\vec{x})$

isotropic: χ is independent of direction

(i.e. could be $\vec{P}_i = \varepsilon_0 \sum_j \chi_{ij} E_j$ ...)

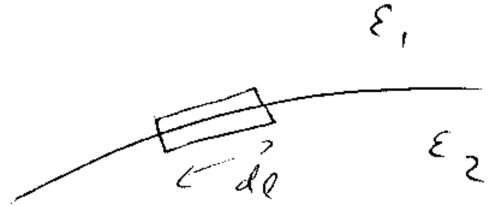
Def χ ~ electric susceptibility

$$\vec{D} = \varepsilon_0\vec{E} + \varepsilon_0\chi\vec{E} = \varepsilon_0(1+\chi)\vec{E} \equiv \varepsilon\vec{E}$$

Def $\varepsilon = \varepsilon_0(1+\chi)$ ~ dielectric constant.

(in vacuum, $\chi = 0$ and $\varepsilon = \varepsilon_0$)

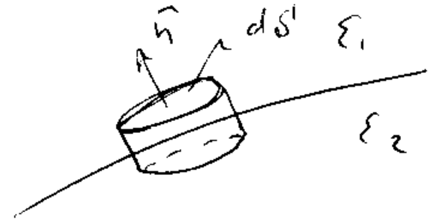
Boundary matching



$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow$$

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \quad \Rightarrow \quad \left(\vec{E}_1 - \vec{E}_2\right) \cdot \hat{t}\, dl = 0$$

$$\Rightarrow \boxed{E_{1t} = E_{2t}}$$



$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad \Rightarrow$$

$$\int_V \vec{\nabla} \cdot \vec{D}\, d^3x = \int \rho_f\, d^3x = \sigma_f \, dS$$

$$\Rightarrow \boxed{D_{1n} - D_{2n} = \sigma_f}$$

$\sigma_f \sim$ surface free charge density.

As we showed before, $E_{1n} - E_{2n} = \dfrac{\sigma}{\varepsilon_0}$

where $\sigma = \sigma_{free} + \sigma_{bound}$ ~ total surface charge density.

as $\vec{E} = \dfrac{1}{\varepsilon_0}\left(\vec{D} - \vec{P}\right) \Rightarrow$

$$E_{1n} - E_{2n} = \frac{1}{\varepsilon_0}\left(D_{1n} - D_{2n} - P_{1n} + P_{2n}\right) = \frac{\sigma_f}{\varepsilon_0} - \frac{P_{1n} - P_{2n}}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

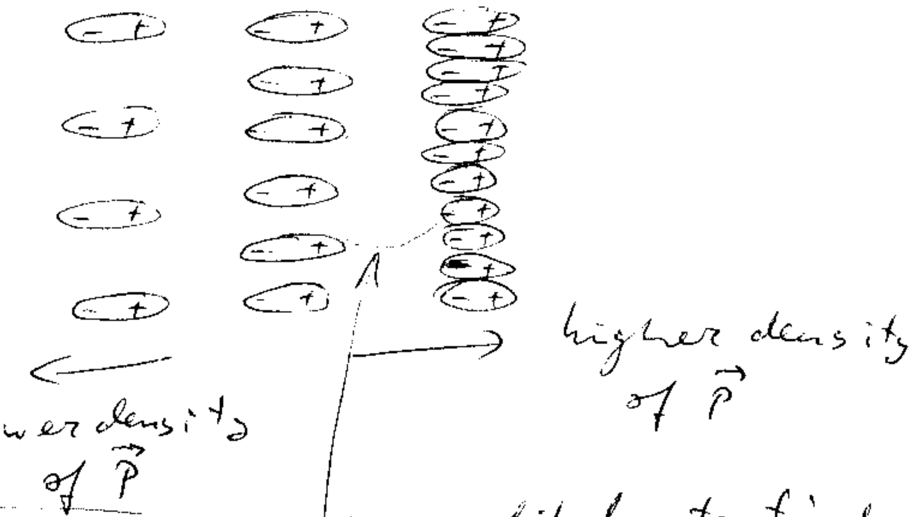
$$\Rightarrow \left\{ P_{1n} - P_{2n} = \sigma_f - \sigma = -\sigma_b \right\}$$

As the bound charge density is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow P_{1n} - P_{2n} = -\sigma_b$.

Why is $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

Pictorially:



more likely to find more negative charges in a volume element

$\Downarrow$

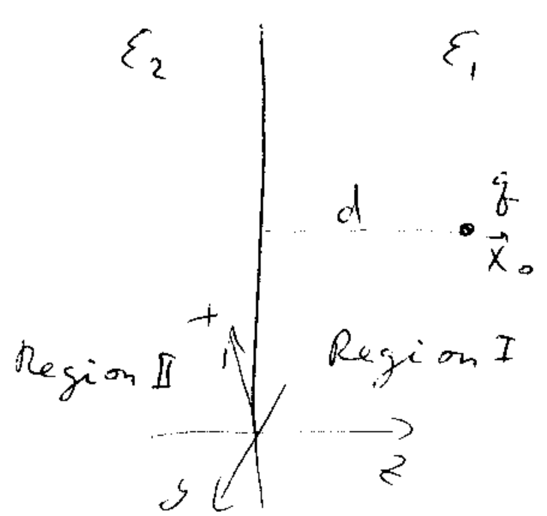
generates ρ_b!

Finally, if $\vec{D} = \varepsilon\vec{E}$

$\Rightarrow \vec{\nabla}\cdot\vec{D} = \varepsilon\vec{\nabla}\cdot\vec{E} = \rho_f \Rightarrow$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon}$$

Boundary-Value Problems with Dielectrics.



Region I: $\vec{\nabla}\cdot\vec{D} = \rho_f = \varepsilon_1\vec{\nabla}\cdot\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon_1} = \frac{q}{\varepsilon_1}\delta^3(\vec{x}-\vec{x}_0)$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi.$$

$\vec{x}_0 = (0, 0, d)$