Electrostatic Energy in Dielectrics.

$W = \frac{1}{2}\int d^3x \, \rho_f \, \Phi$ ~ doesn't apply anymore

As we bring in charges from ∞, we also need to construct correct polarization of the medium.
(effect of ρ_b)

Change $\rho(\vec{x})$ by small quantity $\delta\rho(\vec{x}) \Rightarrow$

$$\delta W = \int_V d^3x \; \delta\rho(\vec{x}) \, \Phi(\vec{x})$$

As $\rho = \vec{\nabla}\cdot\vec{D} \Rightarrow \delta\rho = \vec{\nabla}(\delta\vec{D})$

$$\Rightarrow \delta W = \int_V d^3x \; \vec{\nabla}(\delta\vec{D})\Phi = (\text{parts}) = -\int d^3x \, \delta\vec{D}\cdot$$

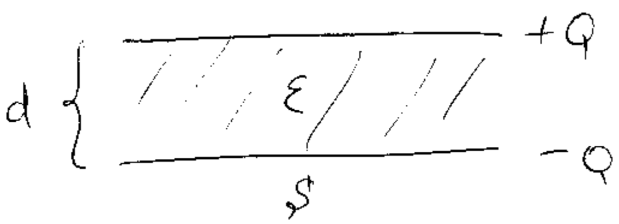
$$\cdot\vec{\nabla}\Phi = \int d^3x \; \vec{E}\cdot\delta\vec{D} \Rightarrow \boxed{W = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}}$$

If medium is linear and isotropic, then $\vec{D}_{(x)} = \varepsilon_{(x)}\vec{E}_{(x)}$

$$\Rightarrow \delta W = \int d^3x \; \vec{E}\cdot\varepsilon\cdot\delta\vec{E} = \delta\left(\int d^3x \; \varepsilon\cdot\frac{1}{2}\vec{E}^2\right) =$$

$$= \delta\left(\int d^3x \; \frac{1}{2}\vec{E}\cdot\vec{D}\right) \Rightarrow \boxed{W = \frac{1}{2}\int d^3x \; \vec{E}\cdot\vec{D}}$$

<u>Example</u> capacitor with dielectric in it.



$\vec{\nabla}\cdot\vec{D} = \rho_{free} \Rightarrow D = Q/S$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{free}}{\varepsilon} \Rightarrow E = \frac{1}{\varepsilon}\frac{Q}{S}$

$$W = \frac{1}{2} \underbrace{S \cdot d}_{\text{volume}} \frac{1}{\varepsilon} \frac{Q^2}{S^2} \Rightarrow \boxed{W = \frac{1}{2} \frac{d\, Q^2}{\varepsilon S}}$$

Capacitance $C = \frac{Q}{\Delta V} = \frac{Q}{E \cdot d} = \frac{Q}{\frac{Q}{S} \frac{1}{\varepsilon} \cdot d} = \frac{\varepsilon S}{d}$

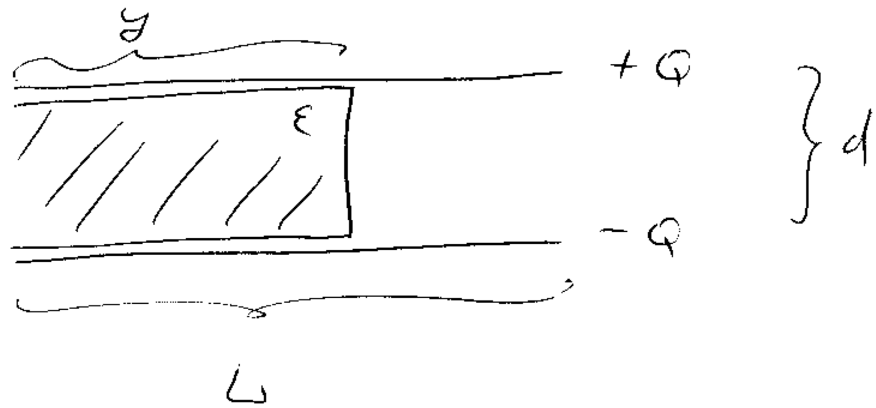
$\Rightarrow \boxed{\frac{C}{S} = \frac{\varepsilon}{d}}$ in vacuum $\varepsilon = \varepsilon_0 \Rightarrow \frac{C}{S} = \frac{\varepsilon_0}{d}$

works!

Forces: $F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q$

Force due to displacement in ξ-direction with sources Q fixed. (insulated from external world)

Example:



$L \times L$ square plates.

In general, surface charge density is different in vacuum & dielectric parts:

$\sigma_d = \varepsilon E_d, \quad \sigma_v = \varepsilon_0 E_v.$