$$W = \frac{1}{2} \underbrace{S \cdot d}_{\text{volume}} \frac{1}{\varepsilon} \frac{Q^2}{S^2} \Rightarrow \boxed{W = \frac{1}{2} \frac{d\,Q^2}{\varepsilon S}}$$

Capacitance $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{E \cdot d} = \dfrac{Q}{\frac{Q}{S}\frac{1}{\varepsilon} \cdot d} = \dfrac{\varepsilon S}{d}$

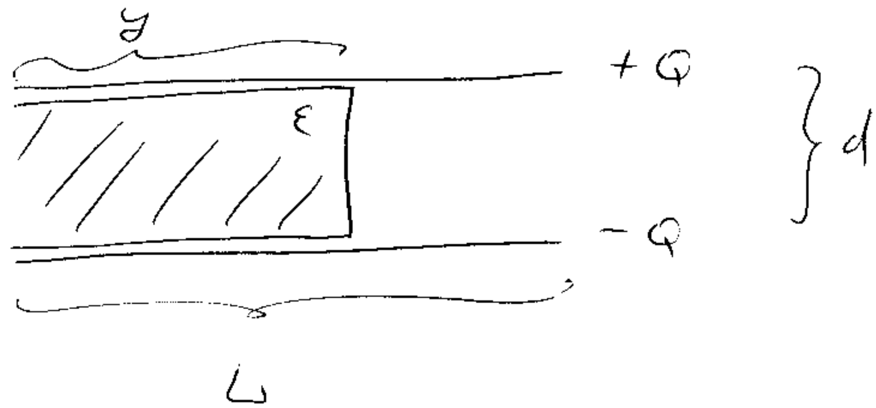
$\Rightarrow \boxed{\dfrac{C}{S} = \dfrac{\varepsilon}{d}}$ in vacuum $\varepsilon = \varepsilon_0 \Rightarrow \dfrac{C}{S} = \dfrac{\varepsilon_0}{d}$

works!

Forces: $F_\xi = -\left(\dfrac{\partial W}{\partial \xi}\right)_Q$

Force due to displacement in ξ-direction with sources Q fixed. (insulated from external world)

Example:



$L \times L$ square plates.

In general, surface charge density is different in vacuum & dielectric parts:

$\sigma_d = \varepsilon E_d$, $\sigma_v = \varepsilon_0 E_v$.

at the interface $E_{d,t} = E_{v,t} \Rightarrow E_d = E_v \equiv E$

$$\Rightarrow Q = L y \sigma_d + L \cdot (L-y) \sigma_v = L (y \cdot \varepsilon + (L-y)\varepsilon_0) E$$

$$\Rightarrow E = \frac{Q}{L[\varepsilon y + \varepsilon_0 (L-y)]}$$

in dielectric $D = \varepsilon E$, in vacuum $D = \varepsilon_0 E$

$\Rightarrow$ total energy $W = \frac{1}{2} \cdot d y L \; D_d \cdot E_d +$

$$+ \frac{1}{2} d (L-y) L \; D_v \, E_v = \frac{1}{2} d y L \cdot \varepsilon \cdot \left(\frac{Q}{L[\varepsilon y + \varepsilon_0 (L-y)]} \right)^2 +$$

$$+ \frac{1}{2} d (L-y) L \cdot \varepsilon_0 \left(\frac{Q}{L[\varepsilon y + \varepsilon_0 (L-y)]} \right)^2 =$$

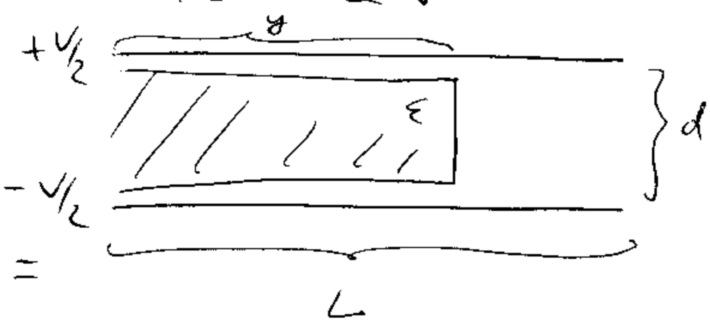
$$= \frac{1}{2} d \frac{Q^2}{L[\varepsilon y + \varepsilon_0 (L-y)]} \Rightarrow F = -\left(\frac{\partial W}{\partial y} \right)_Q \Rightarrow$$

$$\Rightarrow \frac{1}{2} \frac{d Q^2 (\varepsilon - \varepsilon_0)}{L[\varepsilon y + \varepsilon_0 (L-y)]^2} = F > 0$$

$F > 0 \Rightarrow$ the force pulls the slab inside the capacitor

if $\varepsilon = \varepsilon_0 \Rightarrow F = 0$ no force in vacuum. ok.

The problem is different if capacitor plates are held at constant potential difference V:



$$V = E \cdot d \Rightarrow E = \frac{V}{d}$$

$$W = \frac{1}{2} L d \; E^2 \left[\varepsilon y + \varepsilon_0 (L-y) \right] =$$

$$= \frac{1}{2} \frac{LV^2}{d} \left[\varepsilon y + \varepsilon_0 (L-y) \right]$$

$$\Rightarrow F = + \left(\frac{\partial W}{\partial y} \right)_V = \frac{1}{2} \frac{LV^2}{d} (\varepsilon - \varepsilon_0) > 0$$

force still pulls dielectric in

note the sign! The system is not isolated anymore.

When we move the dielectric, we first fix the charges $\Rightarrow \delta W_1 = \frac{1}{2} \int \rho \, \delta \Phi_1 \, d^3x$.

Then we let the charges exit/enter the system to keep potential constant

$$\delta W_2 = \frac{1}{2} \int d^3x \left[\rho \, \delta \Phi_2 + \Phi \, \delta \rho_2 \right]$$

Now, to keep Φ constant, we need $\delta \Phi_1 = - \delta \Phi_2$, $\Rightarrow$

$\delta W_2 = - \delta W_1 + \frac{1}{2} \int d^3x \; \Phi \, \delta \rho_2$. Now, as $\nabla^2 \Phi = - \frac{\rho}{\varepsilon}$ $\Rightarrow$

$\Rightarrow$ both terms in δW_2 are equal $\Rightarrow \delta W_2 = -2 \delta W_1$

$$\Rightarrow \delta W_V = \delta W_1 + \delta W_2 = - \delta W_1 = - \delta W_Q \Rightarrow F = \left(\frac{\partial W}{\partial \xi} \right)_V .$$