Magnetostatics

$\Rightarrow$ main difference from electrostatics is due to absence of magnetic monopoles (no equivalent of point charges).

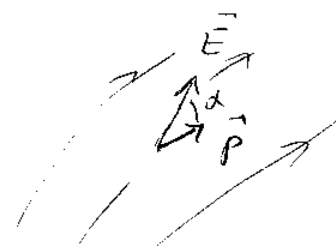
Instead one deals with magnetic dipoles:

loop of current carries magnetic dipole $\vec{m}$

Torque on $\vec{m}$ is $\vec{N} = \vec{m} \times \vec{B}$, where

$\vec{B}$ is <u>magnetic induction</u> (aka magnetic flux density)

(Analogy to electric dipoles: if we have dipole $\vec{p}$ in electric field $\vec{E}$:



$$W = -\vec{p} \cdot \vec{E} =$$
$$= -p\,E \cdot \cos\alpha$$

torque is

$$N = +\frac{\partial W}{\partial \alpha} = p\,E\,\sin\alpha \Rightarrow \vec{N} = \vec{p} \times \vec{E}\ \text{.})$$

<u>Continuity</u> if $\rho(\vec{x},t)$ is charge density and $\vec{J}(\vec{x},t)$ is current density

$$\left(\rho = \frac{\text{charge}}{\text{volume}}\ ,\ J = \frac{\text{charge} \cdot \text{velocity}}{\text{volume}}\right)$$

$\Rightarrow$ the change in total charge in enclosed volume V should be equal to the amount of charge that flowed in/out of the volume. Hence:

$$\Delta Q = \int_V d^3x \left[\rho(\vec{x}, t+\Delta t) - \rho(\vec{x}, t) \right] = -\Delta t \oint_S da \cdot J_n$$

$$= -\Delta t \int_V d^3x \; \vec{\nabla} \cdot \vec{J}$$

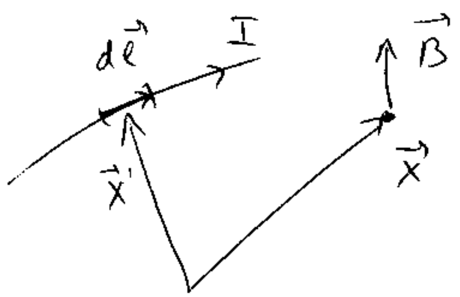
divergence theorem

$\Rightarrow$ as $\rho(\vec{x}, t+\Delta t) - \rho(\vec{x}, t) \approx \Delta t \cdot \frac{\partial \rho}{\partial t}(\vec{x}, t)$

$$\Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

in the static case $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = 0}$.

Biot and Savart Law



$$d\vec{B} = \frac{\mu_0}{4\pi} I \, \frac{d\vec{\ell} \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

$\frac{\mu_0}{4\pi} = 10^{-7} \frac{\text{Newtons}}{\text{Ampere}^2}$, 1 Ampere = 1 Coulomb / 1 Second for a point charge q moving with velocity $\vec{v}$:

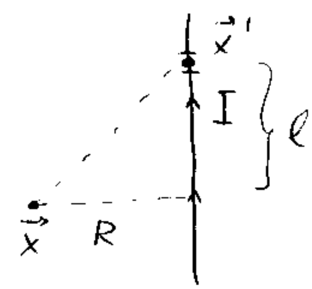
$$I\, d\vec{\ell} = q\,\vec{v} \Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\, q\, \frac{\vec{v} \times \vec{x}}{x^3}.$$

Transforming current I into current density $\vec{J}$ via $I\, d\vec{\ell} = \vec{J} \cdot d^3x$ we write

$$\vec{B} = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$

This is magnetic induction due to any current density $\vec{J}(\vec{x})$.

<u>Example</u>: a wire carrying current I:



$$|\vec{B}| = \frac{\mu_0}{4\pi} I \cdot \int_{-\infty}^{\infty} \frac{d\ell}{\ell^2 + R^2} \cdot \underbrace{\frac{R}{\sqrt{R^2 + \ell^2}}}_{\text{sin of the angle between } d\vec{\ell} \;\&\; (\vec{x} - \vec{x}')} =$$

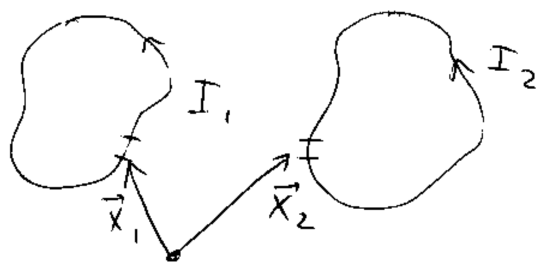
$$= \frac{\mu_0}{4\pi} I R \int_{-\infty}^{\infty} \frac{d\ell}{(\ell^2 + R^2)^{3/2}} = \frac{\mu_0}{2\pi} \frac{I}{R}.$$

Ampere's Law

The force on a current element $I_1 d\vec{l}_1$ due to magnetic field $\vec{B}$ is $\boxed{d\vec{F} = I_1\, d\vec{l}_1 \times \vec{B}}$

For a point charge q moving with velocity $\vec{v}$ write $\vec{F} = q\,\vec{v}\times\vec{B}$ (Lorentz force)

Imagine two loops of current: the force on loop #1 due to loop #2 is



$$\vec{F}_{12} = I_1 \int d\vec{l}_1 \times \vec{B}_2$$

Due to Biot & Savart law, $\vec{B}_2 = \frac{\mu_0}{4\pi} I_2 \int \frac{d\vec{l}_2 \times \vec{x}_{12}}{x_{12}^3}$

where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. Substituting:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3}$$

as $\frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3} = \frac{d\vec{l}_2\,(d\vec{l}_1 \cdot \vec{x}_{12})}{x_{12}^3} - \vec{x}_{12}\frac{d\vec{l}_1 \cdot d\vec{l}_2}{x_{12}^3}$

and, since $\vec{\nabla}_1 \frac{1}{|\vec{x}_{12}|} = -\frac{\vec{x}_{12}}{|x_{12}|^3}$, the first term vanishes and we write:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_{12}|^3} \vec{x}_{12}$$

attractive if $I_1 \parallel I_2$
repulsive if I_1 & I_2 anti-parallel

Ampere's law of force between two current loops.

As $I\, d\vec{l} = \vec{J}\, d^3x \Rightarrow$ for two localized current densities $\vec{F}_{12} = -\frac{\mu_0}{4\pi} \int d^3x_1\, d^3x_2 \frac{\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)}{|\vec{x}_{12}|^3} \vec{x}_{12}$.

For current density Ampere's law gives:

$$\boxed{\vec{F} = \int d^3x\ \vec{J}(\vec{x}) \times \vec{B}(\vec{x})}$$

The resulting torque is $\boxed{\vec{N} = \int d^3x\ \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}(\vec{x}))}$.

Differential Equations of Magnetostatics

Start with Biot & Savart law.

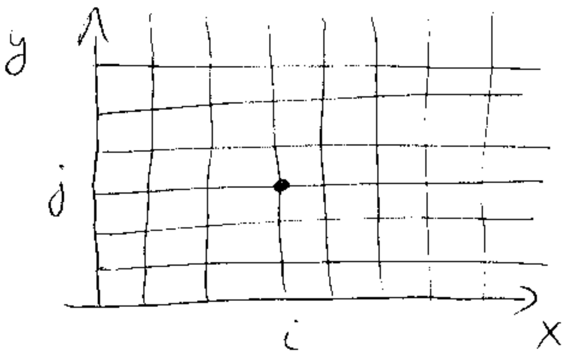
$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = -\frac{\mu_0}{4\pi} \int d^3x' \cdot$$

$$\cdot \vec{J}(\vec{x}') \times \vec{\nabla}_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$\Rightarrow$ we recast $\vec{B}$ as a curl of some vector field.

Numerical Solution of Laplace Equation

~ Relaxation method



$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

$$\frac{\partial \Phi}{\partial x} \to \frac{\Phi(i+1,j) - \Phi(i,j)}{\Delta x}$$

Δx ← lattice spacing

$$\Phi \to \Phi(i,j)$$

$$\left.\begin{aligned}\frac{\partial^2 \Phi}{\partial x^2} &= \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{\Delta x^2} \\ \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{\Delta y^2}\end{aligned}\right\} \text{put } \Delta x = \Delta y = a$$

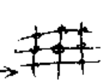
$$\Rightarrow \nabla^2 \Phi = \frac{1}{a^2}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) +$$

$$+ \Phi(i,j-1) - 4\Phi(i,j)\Big] = 0$$

$$\Rightarrow \Phi(i,j) = \frac{1}{4}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + \Phi(i,j-1)\Big]$$

Algorithm: (1) Assign random values to Φ on a grid, with Φ on the boundary given by boundary conditions

(2) average over nearest neighbours until you converge to the answer



(Improvements: include diagonal neighbours, overrelaxation, etc.)