and we write:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_{12}|^3} \vec{x}_{12}$$

attractive if $I_1 \parallel I_2$
repulsive if I_1 & I_2 anti-parallel

Ampere's law of force between two current loops.

As $I\,d\vec{l} = \vec{J}\,d^3x \Rightarrow$ for two localized current densities $\vec{F}_{12} = -\frac{\mu_0}{4\pi}\int d^3x_1\, d^3x_2 \, \frac{\vec{J}_1(\vec{x}_1)\cdot\vec{J}_2(\vec{x}_2)}{|\vec{x}_{12}|^3}\vec{x}_{12}$.

For current density Ampere's law gives:

$$\boxed{\vec{F} = \int d^3x \; \vec{J}(\vec{x}) \times \vec{B}(\vec{x})}$$

The resulting torque is $\boxed{\vec{N} = \int d^3x \; \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}(\vec{x}))}$.

Differential Equations of Magnetostatics

Start with Biot & Savart law.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \; \frac{\vec{J}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = -\frac{\mu_0}{4\pi}\int d^3x' \; \vec{J}(\vec{x}') \times \vec{\nabla}_x \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int d^3x' \; \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ we recast $\vec{B}$ as a curl of some vector field.

Definition: Vector potential $\vec{A}$ is defined by

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

(Analogue of electrostatic potential Φ with $\vec{E} = -\vec{\nabla}\Phi$)

$\vec{A}$ is not observable directly (in classical physics)

$\vec{B}$ is observable

We have the freedom of redefining

$$\vec{A}(\vec{x}) \rightarrow \vec{A}(\vec{x}) + \vec{\nabla}\psi(\vec{x})$$

for any random scalar function $\psi(\vec{x})$:

as $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$, $\vec{B}$ does not change $\Rightarrow$

$\Rightarrow$ gauge invariance!

(cf. $\Phi \rightarrow \Phi + \text{const}$ in electrostatics)

$\vec{A}$ is defined up to a gradient.

Biot-Savart law gives us

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{\nabla}\psi$$

As $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$ (analogy of $\vec{\nabla} \times \vec{E} = 0$)

$\Rightarrow$ no magnetic monopoles & no point sources of $\vec{B}$

On the other hand,

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} =$$

$$= \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \int d^3x' \, \vec{J}(\vec{x}') \cdot$$

$$\cdot \underbrace{\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|}}_{-\vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|} \text{ \& do parts}} - \frac{\mu_0}{4\pi} \int d^3x' \, \vec{J}(\vec{x}') \underbrace{\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|}}_{-4\pi \delta^3(\vec{x} - \vec{x}')} =$$

$$= \mu_0 \vec{J}(\vec{x}) + \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \underbrace{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}_{0}$$

(continuity equation is steady state)

$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$ (analogy of $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$)

To derive an analogy of Gauss's law, integrate

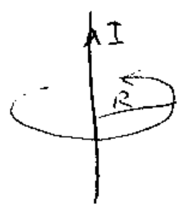
$$\int_S da \; \hat{n} \cdot (\vec{\nabla} \times \vec{B}) = \oint_C \vec{B} \, d\vec{\ell}$$

$$\mu_0 \int_S da \; \hat{n} \cdot \vec{J} \Rightarrow \boxed{\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S da \; \hat{n} \cdot \vec{J} = \mu_0 I}$$

Ampere's law

I ~ total current through the loop.

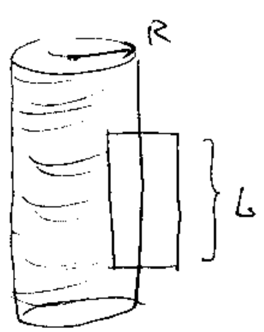
<u>Example</u>: find $\vec{B}$ of a straight wire carrying current I:



$$B \cdot 2\pi R = \mu_0 I \Rightarrow B = \frac{\mu_0}{2\pi} \frac{I}{R}$$

(cf. with what we found using Biot-Savart law earlier)

<u>Example</u>: infinite solenoid, N coils per unit length:



$$B_{in} \cdot L = \mu_0 I \cdot N \cdot L \Rightarrow B_{in} = \mu_0 I N$$

uniform magnetic field inside!

$B_{out} = 0$.

Finally, we know that $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$ & $\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$