Writing $\vec{B} = \vec{\nabla} \times \vec{A}$ automatically satisfies the first equation. The second yields:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}}$$

As we have gauge freedom $\vec{A} \to \vec{A} + \vec{\nabla} \psi$,

$\Rightarrow$ can demand that in new gauge $\vec{\nabla} \cdot \vec{A}_{new} = 0$

$$\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla} \cdot \psi \Rightarrow \vec{\nabla} \cdot \vec{A}_{new} = \vec{\nabla} \cdot \vec{A}_{old} + \vec{\nabla}^2 \psi = 0$$

$\Rightarrow \nabla^2 \psi = - \vec{\nabla} \cdot \vec{A}_{old}$ $\Rightarrow$ can always find ψ by solving this Poisson-like equation

$\vec{\nabla} \cdot \vec{A} = 0$ ~ Coulomb gauge condition.

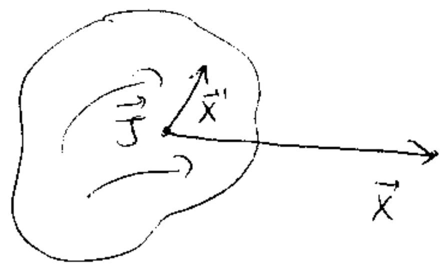
in Coulomb gauge $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad , \quad (\psi = \text{const}).$$

Magnetic Fields of a Localized Current Distribution:

Magnetic Moment.

Imagine a localized current distribution:



We need to find vector potential far away from the currents: start with

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ to properly expand $\vec{A}(\vec{x})$ in powers of $\frac{1}{r}$ we need vector spherical harmonics ~ we'll maybe talk about them next quarter.

$\Rightarrow$ Instead expand

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots$$

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{1}{|\vec{x}|}\int d^3x' \, J_i(\vec{x}') + \frac{\mu_0}{4\pi}\,\frac{\vec{x}\,\cdot}{|\vec{x}|^3}\int d^3x' .$$

$$\cdot\,\vec{x}' \, J_i(\vec{x}') + \ldots$$

Now, $\displaystyle\int d^3x' \, J_i(\vec{x}') = \int d^3x' \left[\vec{\nabla}'\cdot\left(x_i' \, \vec{J}(\vec{x}')\right) - x_i' \, \vec{\nabla}\cdot\vec{J}\right]$

First term becomes a surface integral

$$\int_S da \; x_i J_n = 0 \quad \text{as current is localized}$$

Second term is also $\emptyset$ as $\vec{\nabla}\cdot\vec{J} = 0$ (continuity)

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{x}}{|\vec{x}|^3} \cdot \int d^3x' \, \vec{x}' \, J_i(\vec{x}')$$

as $\vec{\nabla}\cdot\vec{J} = 0$.

Now, $0 = \int d^3x' \, \vec{\nabla}' \cdot \left(x_i' x_j' \vec{J}(\vec{x}') \right) = \int d^3x' \left[x_i' J_j + \right.$

$\left. + x_j' J_i \right] \Rightarrow \int d^3x' \left[x_i' J_j + x_j' J_i \right] = 0$

$$\Rightarrow \vec{x} \cdot \int d^3x' \, \vec{x}' \cdot J_i(\vec{x}') \equiv \sum_j x_j \int d^3x' \, x_j' J_i =$$

$$= -\frac{1}{2} \sum_j x_j \int d^3x' \left[x_i' J_j - x_j' J_i \right] =$$

$$= -\frac{1}{2} \sum_{j,k} \varepsilon_{ijk} x_j \int d^3x' \left(\vec{x}' \times \vec{J} \right)_k$$

as $(\vec{x}' \times \vec{J})_k = \varepsilon_{ki'j'} x'_{i'} J_{j'}$ and

$$\varepsilon_{ijk} \varepsilon_{i'j'k} = \delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}.$$

Finally we obtain $-\frac{1}{2} \left[\vec{x} \times \int d^3x' \left(\vec{x}' \times \vec{J} \right) \right]_i$, Such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left(-\frac{1}{2}\right)\frac{\vec{x}}{|\vec{x}|^3} \times \int d^3x'\ \vec{x}' \times \vec{J}$$

Definition. Defining <u>magnetic moment</u>

$$\vec{m} = \frac{1}{2}\int d^3x'\ \ \vec{x}' \times \vec{J}(\vec{x}')$$

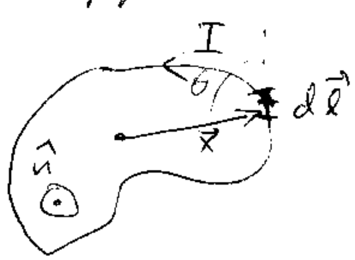
We obtain $\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\ \frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\ \frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}}$$ (cf. with $\vec{E}$ of a dipole)

Definition $\vec{M} = \frac{1}{2}\ \vec{x}\times\vec{J}(\vec{x})$ is the magnetic moment density, or, <u>magnetization</u>

(More precisely, $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta^3(\vec{x})\right]$)

Suppose the current is confined to a plane:



$$\vec{m} = \frac{1}{2}\ I\int\ \vec{x}\times d\vec{\ell}$$

$\Rightarrow$ as $|\vec{x}\times d\vec{\ell}|\cdot\frac{1}{2} = \frac{1}{2}\ x\ d\ell\cdot\sin\theta = \underbrace{da}_{\text{area element}}$

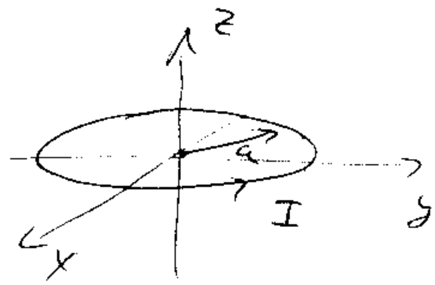
$$\Rightarrow \left| \frac{1}{2} \int \vec{x} \times d\vec{l} \right| = S \quad \text{(area of the loop)}$$

$$\Rightarrow |\vec{m}| = I \cdot S \quad , \text{or} \quad \boxed{\vec{m} = I S \hat{n}}$$

$\hat{n}$ is pointing out of the plane

$\vec{m}$ is independent of origin. Can you prove that?

<u>Example</u>: current loop:



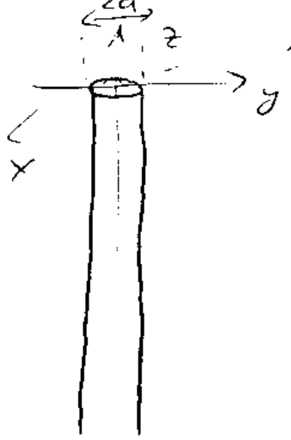
$$\Rightarrow \vec{m} = I \cdot \pi a^2 \cdot \hat{n} = I \pi a^2 \hat{z}.$$

far from the loop.

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0 I a^2}{4} \cdot \frac{\hat{z} \times \vec{x}}{\vec{x}^3} \Rightarrow \text{in spherical coordinates}$$

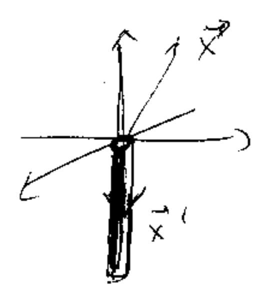
$$A_\varphi = \frac{\mu_0 I a^2}{4} \frac{\sin\theta}{r^2} \quad , \quad A_\theta = A_r = 0.$$

<u>Example</u> consider a half-infinite ideal solenoid; it has current I and N loops per unit length. Each loop carries magn. moment



$\Delta\vec{m} = I \pi a^2 \hat{z}$ Assume that a is tiny $\Rightarrow$

$$\Rightarrow \vec{A}_{(\vec{x})} \simeq \frac{\mu_0}{4\pi} \int \frac{d\vec{m}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}.$$



where $dm = I \pi a^2 N dz$.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\pi a^2 I N \int_{-\infty}^{0} dz' \; \frac{\hat{z}\times(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3} =$$

$$= -\frac{\mu_0}{4\pi}\cdot\pi a^2 I N \int_{-\infty}^{0} dz' \; \hat{z}\times\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}\,'|}$$

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A} = -\frac{\mu_0}{4} a^2 I N \int_{-\infty}^{0} dz' \; \vec{\nabla}\times\left(\hat{z}\times\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}\,'|}\right) =$$

$$= -\frac{\mu_0}{4} a^2 I N \int_{-\infty}^{0} dz' \left[\hat{z}\,\underbrace{\nabla^2\frac{1}{|\vec{x}-\vec{x}\,'|}}_{=\,-4\pi\,\delta^3(\vec{x}-\vec{x}\,')} - \vec{\nabla}\,\underbrace{\frac{\partial}{\partial z}}_{=\,-\partial/\partial z'}\frac{1}{|\vec{x}-\vec{x}\,'|}\right] =$$

$$= \mu_0\,\pi a^2 I N\,\hat{z}\;\delta(x)\,\delta(y)\,\Theta(-z) -$$

$$-\frac{\mu_0}{4} a^2 I N\;\vec{\nabla}\frac{1}{|\vec{x}|}$$

field inside the solenoid, $\mu_0 I N = B$

$$\Rightarrow \boxed{\vec{B} = \mu_0\pi a^2 I N\;\hat{z}\;\delta(x)\,\delta(y)\,\Theta(-z) + \frac{\mu_0}{4} a^2 I N\,\frac{\vec{x}}{|\vec{x}|^3}}$$

field of a point magnetic charge $g = \dfrac{\mu_0 a^2 I N}{4}$

Dirac "monopole": take $a \to 0$ keeping $g \propto a^2 I N$ finite

$$\vec{B} = g\,\frac{\vec{x}}{|\vec{x}|^3}$$

$\Rightarrow$ Independence of the solenoid's shape imposed by quantum mechanics.