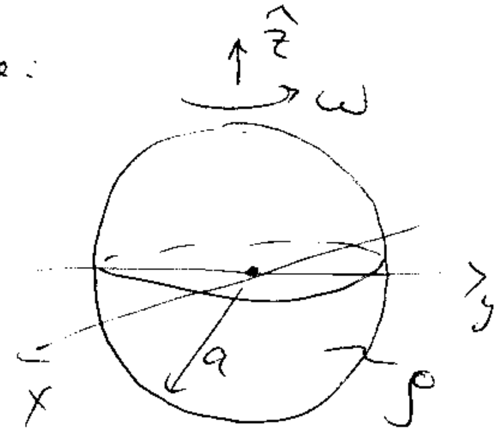
Example: find magnetic dipole moment of a rotating charged sphere:



$$\vec{m} = \frac{1}{2}\int d^3x \quad \vec{x}\times\vec{J}$$

$$\vec{J} = \rho\cdot\vec{v} = \rho\,\vec{\omega}\times\vec{x}$$

$$\Rightarrow \vec{m} = \frac{\rho}{2}\int d^3x \quad \vec{x}\times(\vec{\omega}\times\vec{x}) =$$

$$= \frac{\rho}{2}\int d^3x \left[\vec{\omega}\,|\vec{x}|^2 - \vec{x}\,(\vec{x}\cdot\vec{\omega})\right]$$

$$\Rightarrow \text{as } \vec{\omega} = \omega\,\hat{z} \Rightarrow m_x = m_y = 0$$

$$\Rightarrow m_z = \frac{\rho}{2}\omega\int d^3x\left[r^2 - z^2\right] =$$

$$= \frac{\rho}{2}\,\omega\cdot 2\pi\int_0^a dr\cdot r^2\int_{-1}^{1} d\cos\theta\left[r^2 - r^2\cos^2\theta\right] =$$

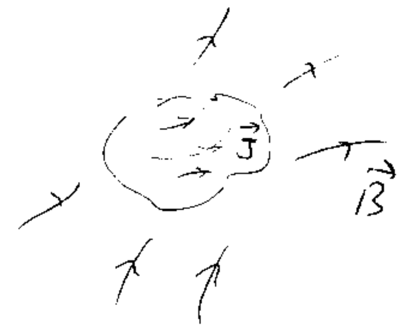
$$= \frac{\rho}{\cancel{2}}\,\omega\,\cancel{2}\pi\,\frac{a^5}{5}\left[2 - \frac{2}{3}\right] = \pi\,\omega\,\rho\,a^5\cdot\frac{4}{15}$$

$$\Rightarrow \text{as } q = \frac{4}{3}\pi a^3\rho \Rightarrow \boxed{m = \frac{1}{5}q\,\omega\,a^2}$$

Torque on $\vec{m}$: $\boxed{\vec{N} = \vec{m}\times\vec{B}(0)}$ (by definition of $\vec{B}$.)

Force on and Energy of a Localized Current.

Consider a system of localized currents in external magnetic induction $\vec{B}$:



$$\vec{F} = \int d^3x \; \vec{J}(\vec{x}) \times \vec{B}(\vec{x})$$

If $\vec{B}$ is slowly varying, write

$$\vec{B}(\vec{x}) = \vec{B}(0) + (\vec{x} \cdot \vec{\nabla})\vec{B}(0) + \dots$$

$$\Rightarrow \vec{F} = \underbrace{\left[\int d^3x \; \vec{J}(\vec{x})\right]}_{\to 0} \times \vec{B}(0) + \int d^3x \; \vec{J}(\vec{x}) \times (\vec{x} \cdot \vec{\nabla})\vec{B}(0)$$

$$\Rightarrow F_i = \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \cdot (\vec{x} \cdot \vec{\nabla}) B_k(0) =$$

$$= \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \; x_l \left(\partial_l B_k\right)\Big|_{\vec{x}=0} =$$

$$= \left(\partial_l B_k\right)\Big|_{\vec{x}=0} \varepsilon_{ijk} \int d^3x \; x_l J_j$$

$$\Rightarrow \text{as} \int d^3x \left(x_i J_j + x_j J_i\right) = 0 \quad \Rightarrow$$

$$F_i = \left(\partial_l B_k\right)\Big|_{\vec{x}=0} \varepsilon_{ijk} \underbrace{\frac{1}{2} \int d^3x \left[x_l J_j - x_j J_l\right]}_{\varepsilon_{ljn} \, m_n}$$

(as $m_i = \frac{1}{2} \varepsilon_{ijk} \int d^3x \, x_j J_k$ $\Rightarrow$

$$\Rightarrow \varepsilon_{\ell j n} \cdot m_n = \frac{1}{2} \underbrace{\varepsilon_{\ell j n} \varepsilon_{n j' k}}_{\delta_{\ell j'} \delta_{jk} - \delta_{\ell k} \delta_{j j'}} \int d^3x \, x_{j'} J_k =$$

$$= \frac{1}{2} \int d^3x \left[x_\ell J_j - x_j J_\ell \right] \Big)$$

$$\Rightarrow F_i = (\partial_\ell B_k)\big|_{\vec{x}=0} \, \varepsilon_{ijk} \varepsilon_{\ell j n} m_n =$$

$$= (\partial_\ell B_k)\big|_{\vec{x}=0} \, m_n \cdot \left[\delta_{i\ell} \delta_{kn} - \delta_{in} \delta_{k\ell} \right] =$$

$$= m_k (\partial_i B_k)\big|_{\vec{x}=0} - m_i \underbrace{(\partial_k B_k)\big|_{\vec{x}=0}}_{0 \text{ as } \vec{\nabla} \cdot \vec{B} = 0}$$

$$\Rightarrow \boxed{\vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B})}$$

If $\vec{F} = -\vec{\nabla} U$, with U the potential energy, $\Rightarrow$ $\boxed{U = -\vec{m} \cdot \vec{B}}$

tends to align dipoles with the magnetic induction $\vec{B}$.

Macroscopic Equations of Magnetostatics

Similar to electrostatics, let's divide the currents into "free" and "bound".

Bound currents are due to magnetic moments of molecules/atoms in the medium and are described by magnetization $\vec{M}(\vec{x})$.

The resulting vector-potential is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \underbrace{\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}_{\text{free current}} + \underbrace{\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}_{\substack{\text{magnetization/} \\ \text{bound currents}}} \right\}$$

The 2nd term is

$$\int d^3x' \frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \int d^3x' \, \vec{M}(\vec{x}')\times \nabla' \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= (\text{parts}) = \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \vec{\nabla}' \times \vec{M}(\vec{x}'),$$ such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}') + \vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ effective current density due to $\vec{M}$ is

$$\vec{J}_M = \vec{\nabla}\times\vec{M}$$