Macroscopic Equations of Magnetostatics

Similar to electrostatics, let's divide the currents into "free" and "bound".

Bound currents are due to magnetic moments of molecules/atoms in the medium and are described by magnetization $\vec{M}(\vec{x})$.

The resulting vector-potential is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \underbrace{\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}_{\text{free current}} + \underbrace{\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}_{\text{magnetization/ bound currents}} \right\}$$

The 2nd term is

$$\int d^3x' \frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \int d^3x' \, \vec{M}(\vec{x}')\times \nabla' \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$=(\text{parts}) = \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \vec{\nabla}'\times\vec{M}(\vec{x}'),$$ such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')+\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ effective current density due to $\vec{M}$ is

$$\vec{J}_M = \vec{\nabla}\times\vec{M}$$

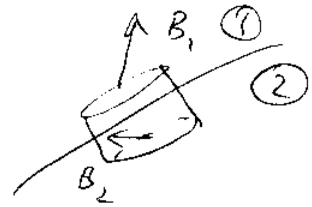
$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \vec{\nabla} \times \vec{M} \right]}$$

Define magnetic field $\boxed{\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}}$

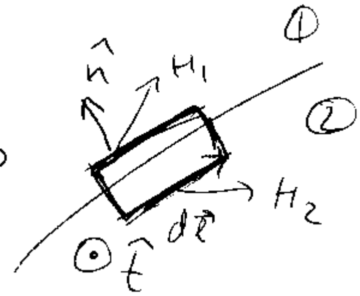
$\Rightarrow$ Macroscopic equations of magnetostatics are $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ and $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

(cf. $\vec{\nabla} \cdot \vec{D} = \rho_f$, $\vec{\nabla} \times \vec{E} = 0$ in electrostatics)

Boundary conditions: $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$



$$\boxed{B_{1n} = B_{2n}}$$

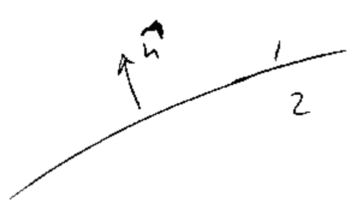


$\Rightarrow$

$\vec{\nabla} \times \vec{H} = \vec{J} \Rightarrow$

$(\vec{H}_1 - \vec{H}_2) \cdot \vec{dl} = \vec{K} \cdot \hat{t}\, dl$, where $\vec{K}$ = surface charge density (amps/m).

$$\Rightarrow \boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$$



At the boundary of two media with magnetizations $\vec{M}_1$ & $\vec{M}_2$: $\vec{J}_M = \vec{\nabla} \times \vec{M}$ and $\hat{n} \times (\vec{H}_1 - \vec{H}_2) =$

$$= \hat{n} \times \left[\frac{1}{\mu_0} \vec{B}_1 - \vec{M}_1 - \frac{1}{\mu_0} \vec{B}_2 + \vec{M}_2 \right] = \vec{K}$$

Therefore $\frac{1}{\mu_0} \hat{n} \times (\vec{B}_1 - \vec{B}_2) = \vec{K} + \vec{K}_M$

where $\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)$ ~ surface current due to magnetization.

What's missing? Relation between $\vec{B}$ & $\vec{H}$!

For linear isotropic homogeneous media

$$\boxed{\vec{B} = \mu \vec{H}}, \quad \mu \text{ is magnetic permeability.}$$

As $\vec{B} = \mu_0 (\vec{H} + \vec{M}) \Rightarrow \vec{M} = \frac{1}{\mu_0}\vec{B} - \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H} =$

$$= \chi_m \vec{H} \Rightarrow \vec{M} = \chi_m \vec{H}, \quad \chi_m = \frac{\mu}{\mu_0} - 1$$

magnetic susceptibility

$\chi_m > 0$ ~ paramagnetic (atoms/molecules have some angular momentum)

$\chi_m < 0$ ~ diamagnetic (no net angular momentum on atoms/molecules ~ resist magnetic field)

$\vec{M} \neq 0$ independent of $\vec{H}$ ~ ferromagnetic

(paramagnetic materials with self-interactions between spins ~ e.g. Ising model)

Solving Boundary-Value Problems in Magnetostatics

(I) No ferromagnetics (no frozen $\vec{M} \neq 0$ for all $\vec{H}$):

A. Vector Potential ($\vec{J} \neq 0$)

$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$ in Coulomb gauge $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

$\Rightarrow$ given $\vec{J}$ can always find $\vec{A}$. (vacuum)

As $\vec{\nabla} \times \vec{H} = \vec{J} = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu}\right) = -\frac{1}{\mu} \nabla^2 \vec{A} \Rightarrow$

in medium $\nabla^2 \vec{A} = -\mu \vec{J}$.

B. $\vec{J} = 0 \Rightarrow$ Magnetic Scalar Potential.

$\vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$

$\Rightarrow$ write $\vec{H} = -\vec{\nabla} \Phi_M \Rightarrow \vec{B} = \mu \vec{H} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

gives $\vec{\nabla} \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Phi_M = 0$ ~ Laplace eqn.

(II) Ferromagnetics ($\vec{M} \neq 0$, $\vec{J} = 0$)

A. Vector potential.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$ as $\vec{\nabla} \times \vec{H} = 0 \Rightarrow$

$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = 0 \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{\nabla} \times \vec{M}$

$\Rightarrow$ without boundaries: $$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

with boundaries:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\mu_0}{4\pi}\oint_S da' \, \frac{\vec{M}(\vec{x}') \times \hat{n}}{|\vec{x} - \vec{x}'|}$$

(Remember ~ $\vec{K} = (\vec{M}_2 - \vec{M}_1) \times \hat{n}$ is the surface current)

B. Magnetic Scalar Potential

$0 = \vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0 \Rightarrow$ defining $\vec{H} = -\vec{\nabla}\Phi_M$

we get $\nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ ~ Poisson-like eqn.

$$\Rightarrow \Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$ no boundaries

With boundaries

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi}\oint_S da' \, \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

magnetic surface-charge density

$$\sigma_M = \hat{n} \cdot \vec{M}.$$