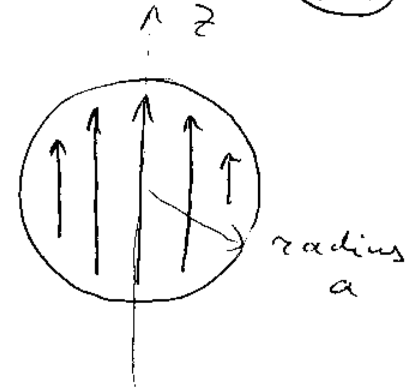
Example: Ferromagnetic sphere:



$\vec{M}$ constant, $\vec{J} = 0$, $\vec{M} = M \cdot \hat{z}$

Find $\vec{B}, \vec{H}$ everywhere

$\Rightarrow$ Use magnetic potential:

$$\vec{\nabla} \cdot \vec{M} = 0 \Rightarrow \Phi_M = \frac{1}{4\pi} \oint_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} =$$

$$= \frac{M}{4\pi} \cdot a^2 \cdot \int d\Omega' \frac{\cos\theta'}{|\vec{x} - \vec{x}'|}$$

Using $\displaystyle \frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$

and $Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$ we get

$$\Phi_M = \frac{Ma^2}{4\pi} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \, 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \cdot$$

$$\cdot Y_{lm}^*(\theta', \varphi') \cdot \sqrt{\frac{4\pi}{3}} Y_{10}(\theta', \varphi') Y_{lm}(\theta, \varphi) = \left(\begin{array}{l}\text{only } l=1 \\ m=0 \text{ is} \neq 0\end{array}\right)$$

$$= Ma^2 \frac{1}{3} \frac{r_<}{r_>^2} \cdot \cos\theta \text{ , where } r_{\lessgtr} = \begin{array}{c}\min \\ \max\end{array} \{ r, a \}$$

$\Rightarrow$ inside $\quad \Phi_M = \frac{1}{3} M r \cos\theta = \frac{1}{3} M z$

outside $\quad \Phi_M = \frac{1}{3} \frac{M a^3}{r^2} \cos\theta$

$$\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M \Rightarrow \vec{H} = -\frac{1}{3}\vec{M} \text{ inside}$$

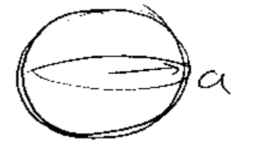
$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\vec{M} \quad \text{inside.}$$

Outside: $\vec{H} = -\vec{\nabla}\left(\frac{1}{3}\frac{Ma^3}{r^2}\cos\theta\right)$; $\vec{B} = \mu_0\vec{H}$.

$$\Rightarrow \vec{B} = -\frac{1}{3}\mu_0 M a^3\, \vec{\nabla}\left(\frac{\hat{z}\cdot\vec{r}}{r^3}\right) \Rightarrow \text{dipole moment}$$

$$\vec{m} = \frac{4\pi a^3}{3}\vec{M}.$$

<u>Example</u>: a sphere with permeable magnetic material ($\vec{B} = \mu\vec{H}$ inside, $\vec{B} = \mu_0\vec{H}$ outside) in external magnetic field $\vec{H}_0$ $\quad\uparrow\uparrow\uparrow\vec{H}_0$



$\nabla^2\Phi_M = 0$ inside & outside

$$\Rightarrow \Phi_{inside} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta), \quad r<a$$

$$\Phi_{outside} = -H_0 r P_1(\cos\theta) + \sum_{\ell=1}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta), \quad r>a$$

as $H_{in,t} = H_{out,t} \Rightarrow$

$\Rightarrow (1)\ \frac{\partial\Phi_1}{\partial\theta} = \frac{\partial\Phi_2}{\partial\theta}$ at $r=a$; $\quad B_{1n} = B_{2n} \Rightarrow \mu B_{in,n} = \mu_0 B_{out,n}$

$$(2)\ \mu\left.\frac{\partial\Phi_{in}}{\partial r}\right|_{r=a} = \mu_0\left.\frac{\partial\Phi_{out}}{\partial r}\right|_{r=a}$$

$$\Rightarrow (1)\quad A_1 a = -H_0 a + \frac{B_1}{a^2}$$

$$(2)\quad \mu A_1 = -\mu_0 H_0 - \frac{2B_1\mu_0}{a^3}$$

$$-H_0 + \frac{B_1}{a^3} = -\frac{\mu_0}{\mu} H_0 - \frac{2B_1}{a^3}\frac{\mu_0}{\mu}$$

$$\Rightarrow \frac{B_1}{a^3}\left(1 + 2\frac{\mu_0}{\mu}\right) = H_0\left(1 - \frac{\mu_0}{\mu}\right) \Rightarrow B_1 = H_0 a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0}$$

$$A_1 = -H_0 + \frac{B_1}{a^3} \Rightarrow A_1 = -\frac{3\mu_0}{\mu + 2\mu_0} H_0$$

$$\Rightarrow \Phi_{inside} = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 r \cos\theta = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 z$$

$$\Phi_{outside} = -H r \cos\theta + H_0 a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0}\frac{1}{r^2}\cos\theta$$

& one can find $\vec{H}_{inside} = -\nabla \Phi_{inside} = \frac{3\mu_0}{\mu + 2\mu_0} \vec{H}_0$

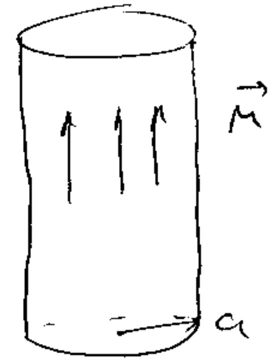
$$\vec{B}_{inside} = \mu \vec{H}_{inside}$$

Effective magnetization: $\vec{M} = \frac{1}{\mu_0}\vec{B}_{inside} - \vec{H}_{inside}$

$= \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}_{inside} \Rightarrow \vec{M} = \frac{3}{\mu_0}\frac{\mu - \mu_0}{\mu + 2\mu_0}\vec{H}_0$ cf. Jackson (5.115)

Example: infinite cylindrical bar magnet:

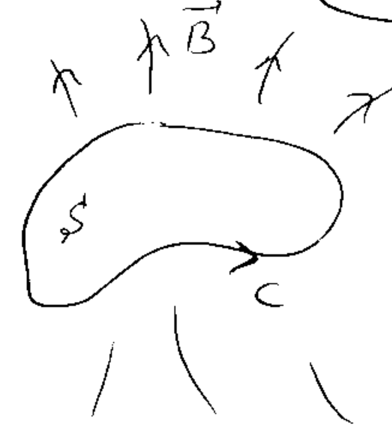
find $\vec{B}$, $\vec{H}$



$$\Phi_M = \frac{1}{4\pi}\oint_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} = 0$$

$$\Rightarrow \vec{H} = 0, \quad \vec{B}_{inside} = \mu_0(\vec{H} + \vec{M}) = \mu_0 \vec{M}, \quad \vec{B}_{outside} = 0.$$

Faraday's Law of Induction.



Suppose we have a current loop C in external magnetic field.

Faraday observed that changes in $\vec{B}$ generate current in the loop.

Namely he observed that, if we define magnetic flux $\Phi = \int_S da\ \vec{B}\cdot\hat{n}$, then

$$\boxed{\mathcal{E} = \oint_C \vec{E}'\cdot d\vec{\ell} = -k\,\frac{d\Phi}{dt}}$$

where $\mathcal{E}$ is called electromotive force.

k is a proportionality constant, $k=1$ in SI units

"$-$" sign is due to Lenz's law ~ the system tries to oppose changes.

$\vec{E}'$ ~ electric field at the element $d\vec{\ell}$ in its rest frame (!)

k is fixed from Galilean invariance:

Φ changes either due to

(i) changes in $\vec{B}$

or

(ii) changes in circuit S ~ motion of C.

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B} = \frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times(\vec{B}\times\vec{v}) + \vec{v}\,\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}$$

$$\Rightarrow \frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,da = \int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,da + \int_S da\;\hat{n}\cdot\vec{\nabla}\times(\vec{B}\times\vec{v}) =$$

$$= \int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,da + \oint_C d\vec{l}\cdot(\vec{B}\times\vec{v})$$

$\Rightarrow$ Faraday's law gives

$$\mathcal{E} = \oint_C \vec{E}'\cdot d\vec{l} = -k\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,da - k\oint_C d\vec{l}\cdot(\vec{B}\times\vec{v})$$

$$\Rightarrow \oint_C d\vec{l}\cdot\left(\vec{E}' - k(\vec{v}\times\vec{B})\right) = -k\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,da$$

This is Faraday's law for a moving circuit.

If we consider the case such that the circuit is at rest, in this situation we have

$$\oint_C \vec{E}\cdot d\vec{l} = -k\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,da$$

$$\Rightarrow \boxed{\vec{E}' = \vec{E} + k\,\vec{v}\times\vec{B}}$$

~ electric field in one frame becomes magnetic induction in another

moving frame — lab frame

as $F = qE$ and $F = qvB \Rightarrow k = 1$.

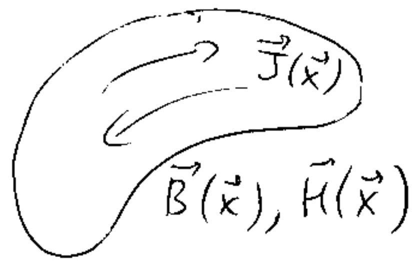
$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

differential form of Faraday's law (generalizes $\vec{\nabla}\times\vec{E} = 0$ of electrostatics).

Energy in the Magnetic Field

Energy change rate is (for a point charge)



$$\frac{dW}{dt} = \vec{V}\cdot\vec{F} = q\,\vec{V}\cdot\vec{E}$$

(point charge q moving with velocity $\vec{v}$)

$$\Rightarrow \delta W = -\delta t\cdot \vec{J}\cdot\vec{E}.$$

Let's change $\vec{B}$ by $\delta\vec{B}(\vec{x})$:

$$\delta W = -\delta t\int d^3x\, \vec{J}\cdot\vec{E} = -\delta t\int d^3x\, (\vec{\nabla}\times\vec{H})\cdot\vec{E}$$

As $\vec{\nabla}(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot\vec{\nabla}\times\vec{H} \Rightarrow$

$$\delta W = +\delta t\int d^3x \left\{ \underbrace{\vec{\nabla}(\vec{E}\times\vec{H})}_{\text{surface integral}} - \vec{H}\cdot\vec{\nabla}\times\vec{E} \right\}$$

$$= -\delta t\int d^3x\, \vec{H}\cdot(\vec{\nabla}\times\vec{E}) \Rightarrow \text{as } \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \boxed{\delta W = \int d^3x \; \vec{H} \cdot \delta \vec{B}}$$

If $\vec{B} = \mu \vec{H} \Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{H} \cdot \vec{B}}$

alternatively, as $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$

$$W = \frac{1}{2} \int d^3x \; \vec{H} \cdot (\vec{\nabla} \times \vec{A}) = \frac{1}{2} \int d^3x \; \vec{A} \cdot \underbrace{(\vec{\nabla} \times \vec{H})}_{= \vec{J}}$$

$$\Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{A} \cdot \vec{J}}$$

Self- and Mutual Inductances.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \Rightarrow$$

$$\Rightarrow W = \frac{\mu_0}{8\pi} \int d^3x \, d^3x' \; \frac{\vec{J}(\vec{x}) \cdot \vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}$$

If we have N circuits with currents $I_1, \ldots, I_N$:

$$W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

Definition $L_i \sim$ self-inductance

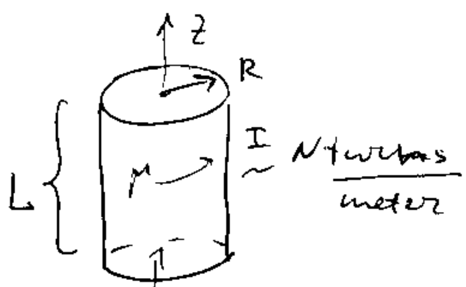
$M_{ij} \sim$ mutual inductance between i & j

as $$W = \frac{\mu_0}{8\pi} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

$$\Rightarrow L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} d^3x_i \int_{V_i} d^3x_i' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

<u>Example</u>: self-inductance of a solenoid:



filled with material with permeability μ.

$$\vec{\nabla} \times \vec{H} = \vec{J} \Rightarrow \vec{H}_{in} = N I \hat{z}, \quad \vec{H}_{out} = 0$$

$$\vec{B}_{in} = \mu \vec{H}_{in} = \mu N I \hat{z}$$

$$\Rightarrow W = \frac{1}{2} \int d^3x \, \vec{B} \cdot \vec{H} = \frac{1}{2} \mu N^2 I^2 \cdot V$$

$V = \pi R^2 \cdot L$ ~ volume

$$\Rightarrow W = \frac{1}{2} L I^2 = \frac{1}{2} \mu N^2 I^2 \pi R^2 L$$

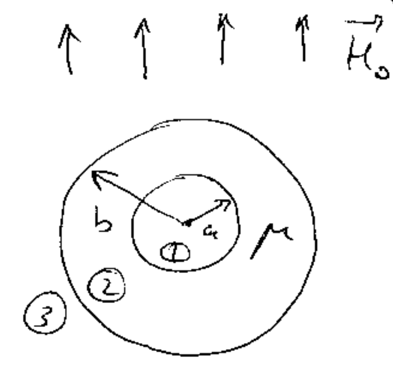
$$\Rightarrow \boxed{L = \mu N^2 \pi R^2 L}$$

Linear currents: $\vec{J} d^3x = I d\vec{l} \Rightarrow W = \frac{1}{2} \int d^3x \, \vec{J} \cdot \vec{A} = \frac{1}{2} \oint \vec{A} \cdot d\vec{l} \cdot I =$

$$= I \cdot \frac{1}{2} \int da \cdot \hat{n} \underbrace{(\vec{\nabla} \times \vec{A})}_{= \vec{B}} = \frac{1}{2} I \underbrace{\Phi}_{\text{magnetic flux}} = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \frac{\Phi}{I}}$$

formula known from undergraduate E&M.

Magnetic Shielding:



$\nabla^2 \Phi_M = 0$ everywhere $\Rightarrow$

$$\Rightarrow \Phi_1 = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta), \quad r<a$$

$$\Phi_2 = \sum_\ell \left(B_\ell r^\ell + C_\ell r^{-\ell-1}\right) P_\ell(\cos\theta), \quad a<r<b$$

$$\Phi_3 = -H_0 r P_1(\cos\theta) + \sum_\ell D_\ell r^{-\ell-1} P_\ell(\cos\theta), \quad r>b$$

Imposing conservation of H_t and B_n at boundaries: $\frac{\partial \Phi_M}{\partial\theta}\begin{pmatrix} b_+ \\ a_+ \end{pmatrix} = \frac{\partial \Phi_M}{\partial\theta}\begin{pmatrix} b_- \\ a_- \end{pmatrix}$

$$\mu_0 \frac{\partial \Phi_1}{\partial r}(r=a) = \mu \frac{\partial \Phi_2}{\partial r}(r=a); \quad \mu \frac{\partial \Phi_2}{\partial r}(r=b) = \mu_0 \frac{\partial \Phi_3}{\partial r}(r=b)$$

Only $\ell = 1$ terms survive:

$$D_1 = \frac{(2\mu+\mu_0)(\mu-\mu_0)}{(2\mu+\mu_0)(\mu+2\mu_0) - 2\frac{a^3}{b^3}(\mu-\mu_0)^2}(b^3-a^3) H_0$$

$$A_1 = -\frac{9\mu\mu_0}{(2\mu+\mu_0)(\mu+2\mu_0) - 2\frac{a^3}{b^3}(\mu-\mu_0)^2} H_0$$

For material with high magnetic permeability

$$P_1 \simeq b^3 H_0 , \quad A_1 \simeq - \frac{9\mu_0}{2\mu\left(1-\frac{a^3}{b^3}\right)} H_0$$

$\Rightarrow A_1 \to 0$ as $\mu \to \infty$ $\Rightarrow \Phi_1 \to 0$ as $\mu \to \infty$ $\Rightarrow$

$\Rightarrow \vec{H}, \vec{B} \to 0$ inside as $\mu \to \infty$ $\Rightarrow$

<u>magnetic screening</u> !

Final: closed books, 2 cheat-sheets allowed (one from midterm, one-new), 5 problems, covers the whole quarter's material, study h/w problems + midterm ⇒ should do fine

→ Dec. 5, 9:30 - 11:18 a.m., Smith 1180