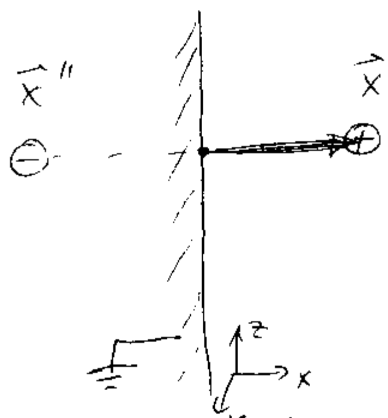
Use the freedom of redefining $G \to G + F$, where $\nabla^2 F = 0$, to fix boundary conditions for $G(\vec{x}, \vec{x}')$.

<u>Example</u>: conductors are equipotential (if not $\Rightarrow$ get $\vec{E} \neq 0 \Rightarrow$ will become equipotential) $\Rightarrow$ natural candidate for Dirichlet boundary conditions ~ conducting surfaces as boundaries



infinite flat conductor

$\Rightarrow$ interested in potential outside conductor

$$G = \frac{1}{|\vec{x} - \vec{x}'|} \quad \text{outside}$$

Can add $F = -\dfrac{1}{|\vec{x} - \vec{x}''|}$ with $(-x', y', z') = \vec{x}''$ as $\nabla^2 F = 0$ in the <u>volume of interest</u>.

One gets $G' = \dfrac{1}{|\vec{x} - \vec{x}'|} - \dfrac{1}{|\vec{x} - \vec{x}''|} \Rightarrow G' = 0$ on the surface

(I) To solve Dirichlet b.c. problem choose

$G_D(\vec{x}, \vec{x}') = 0$ for $\vec{x}'$ on $S \Rightarrow$ using master formula

$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x'\, G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S \phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n} da'}$$

$\Rightarrow$ if one knows boundary condition $\phi(\vec{x})$ on S' and $G_D(\vec{x},\vec{x}')$, along with the charge density $\rho(\vec{x})$ $\Rightarrow$ can find $\phi(\vec{x})$ anywhere in V.

(II) to solve Neumann boundary conditions:

can't just put $\frac{\partial G_N}{\partial n'} = 0$, as due to

$$\nabla^2 G_N(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \int_V d^3\vec{x}'\ \nabla^2 G_N(\vec{x},\vec{x}') = \int_V d^3x'\ \vec{\nabla}\cdot\vec{\nabla} G_N(\vec{x},\vec{x}') =$$

$$= \begin{pmatrix}\text{divergence}\\ \text{thm}\end{pmatrix} = \oint_S da'\ \frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} \underset{\substack{\uparrow\\ \text{if } \vec{x}\in V}}{=} -4\pi \quad \text{due to def. of } G_N$$

$\Rightarrow$ $\frac{\partial G_N}{\partial n'} = 0$ does not work

Instead: $\frac{\partial G_N}{\partial n'} = \frac{-4\pi}{\text{area of } S'} = -\frac{4\pi}{S}$

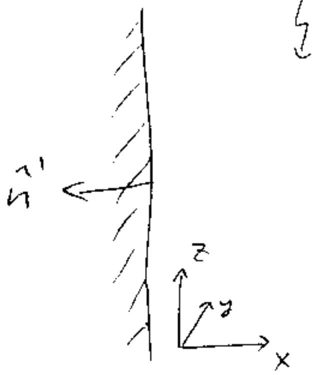
$$\Rightarrow \boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\ G_N(\vec{x},\vec{x}')\,\rho(\vec{x}') + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial\phi}{\partial n'}\,da' + \langle\phi\rangle_{\text{surface}}}$$

where $\langle\phi\rangle_{\text{surface}} \equiv \frac{1}{S}\oint_S \phi(\vec{x}')\,da'$

Again, know ρ, G_N & b.c. $\frac{\partial \phi}{\partial n'}$ $\Rightarrow$ got $\phi(\vec{x})$.

Example: Dirichlet problem:

volume of interest (half-space)



$$\boxed{G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|}}$$

where $\vec{x}' = (x', y', z')$ and $\vec{x}'' = (-x', y', z')$

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, G_D(\vec{x},\vec{x}')\, \rho(\vec{x}') -$$

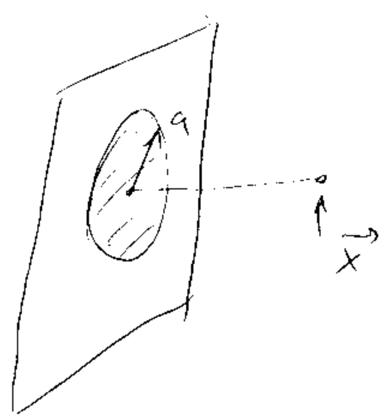
$$- \frac{1}{4\pi} \oint_S \phi(\vec{x}') \frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'} da'$$

$$\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial x'} = \frac{x'-x}{|\vec{x}-\vec{x}'|^3} - \frac{x'+x}{|\vec{x}-\vec{x}''|^3} \Rightarrow$$

$$\Rightarrow \boxed{\left.\frac{\partial G_D}{\partial n'}\right|_{x'=0} = \frac{-2x}{|\vec{x}-(y'\hat{y}+z'\hat{z})|^3}}$$ $\Rightarrow$ know the solution given $\rho(\vec{x})$ & b.c.'s

Suppose $\rho(\vec{x}) = 0$ and $\phi(x,y,z) = \begin{cases} \phi_0, & y^2+z^2 < a^2 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \phi(\vec{x}) = -\frac{\phi_0}{4\pi} \int dy'dz' \frac{-2x}{|\vec{x}-(y'\hat{y}+z'\hat{z})|^3}$$



Suppose $y = z = 0$ (potential along the axis)

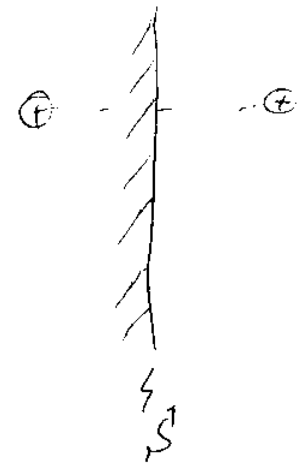
$$\Rightarrow \phi(x,0,0) = \frac{\phi_0 x}{2\pi} \int_0^a 2\pi\rho\, d\rho \frac{1}{[\rho^2+x^2]^{3/2}} \Rightarrow$$

$$\boxed{\phi(x,0,0) = \phi_0\left(1 - \frac{x}{\sqrt{x^2+a^2}}\right)}$$

check: $\phi(0,0,0) = \phi_0$, b.c. is satisfied.

The same problem with Neumann boundary conditions:

need $\dfrac{\partial G_N}{\partial n'} = -\dfrac{4\pi}{S'}$, but since $S' = \infty$



$\Rightarrow \dfrac{\partial G_N}{\partial n'} = 0 \Rightarrow$ like zero electric field

$\Rightarrow$ use "anti-image"

$$\boxed{G_N(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} + \frac{1}{|\vec{x}-\vec{x}\,''|}}$$

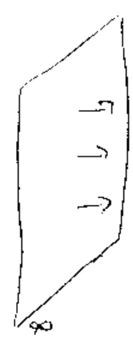
check: $\dfrac{\partial G_N}{\partial n'} = \dfrac{x'-x}{|\vec{x}-\vec{x}\,'|^3} + \dfrac{x+x'}{|\vec{x}-\vec{x}\,''|^3} = 0$ if $x' = 0$

$\quad\quad \overset{\shortparallel}{-\dfrac{\partial G_N}{\partial x'}}$

$$\Rightarrow \phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\, G_N(\vec{x},\vec{x}\,')\rho(\vec{x}\,') + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}\,')\frac{\partial\phi}{\partial n'}da' +$$

$$+\langle\phi\rangle_{surface}$$

If $\rho(\vec{x}) = 0$ and $\left.\dfrac{\partial\phi}{\partial n'}\right|_{x'=0} = E_0$ (const. electric field) $\quad \dfrac{\partial \phi}{\partial n'} = -\dfrac{\partial\phi}{\partial x'}$



E_0

$$\phi(\vec{x}) = \frac{E_0}{2\pi}\int 2\pi\rho\, d\rho\, \frac{1}{(x^2+\rho^2)^{1/2}} = E_0(x^2+\rho^2)^{1/2}\Big|_0^\infty = -E_0 x + \text{const}$$

$\Rightarrow \phi(\vec{x}) = -E_0 x + \langle \phi \rangle_{surface}$.

$\hookrightarrow$ constant electric field everywhere.

Electrostatic Energy

Assume that we're building a system of charges by bringing them in from ∞. Work done to move charge q_i (and its potential energy) is $W_i = q_i \, \Phi(\vec{x}_i)$

where $\Phi(\vec{x}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{i-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$ is the potential due to all charges which have been moved in from ∞ by the time we bring in charge q_i.

After bringing in n charges we have

$$W = \sum_{i=1}^{n} W_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \quad \text{or}$$

$$\boxed{W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}}$$

more symmetric form.

Let's go to continuous limit $\{q_i\} \to \rho(\vec{x})$

$$\Rightarrow \boxed{W = \frac{1}{8\pi\varepsilon_0} \int d^3x \, d^3x' \, \frac{\rho(\vec{x})\,\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}}$$

Watch out: we've just included self-interactions, since $\vec{x} = \vec{x}\,'$ is not excluded!

as $\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$

$$\Rightarrow W = \frac{1}{2} \int d^3x \, \rho(\vec{x}) \, \phi(\vec{x})$$

Using Poisson equation, $\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$

get $W = -\frac{\varepsilon_0}{2} \int d^3x \, \phi(\vec{x}) \, \nabla^2 \phi(\vec{x}) =$ "parts" $=$

$$= -\frac{\varepsilon_0}{2} \int d^3x \left[\underbrace{\vec{\nabla} \left(\phi \, \vec{\nabla} \phi \right)}_{\text{surface integral} \Rightarrow \emptyset} - \left(\vec{\nabla} \phi \right)^2 \right] =$$

$$= \frac{\varepsilon_0}{2} \int d^3x \left(\vec{\nabla} \phi \right)^2 = \Big| \text{ as } \vec{E} = \vec{\nabla} \phi \Rightarrow W = \frac{\varepsilon_0}{2} \int d^3x \, |\vec{E}|^2$$

energy <u>density</u> $w = \frac{\varepsilon_0}{2} \vec{E}^2.$