If $\phi(R,\theta',\varphi') = V$ a constant & $\rho = 0$

$$\Rightarrow \phi(\vec{x}) = \frac{1}{4\pi}\int \underbrace{d\Omega}_{d\varphi\, d\cos\beta} V \frac{R\,(x^2-R^2)}{(x^2+R^2-2xR\cos\beta)^{3/2}} =$$

$$= \frac{1}{4\pi}\cdot 2\pi \cdot \int_{-1}^{1} d\cos\beta \, \frac{R\,(x^2-R^2)}{(x^2+R^2-2xR\cos\beta)^{3/2}}\, V =$$

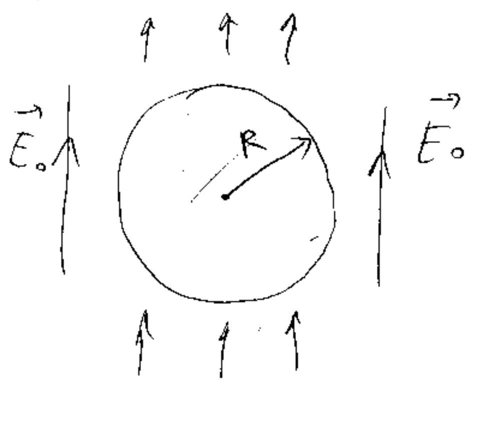
$$= \frac{1}{2}\,\frac{R\,(x^2-R^2)}{xR}\, V \left.\frac{1}{(x^2+R^2-2xR\cos\beta)^{1/2}}\right|_{-1}^{1} =$$

$$= \frac{1}{2}\,\frac{x^2-R^2}{x}\, V\left(\frac{1}{|x-R|} - \frac{1}{|x+R|}\right) = \text{as } x > R =$$

$$= V\,\frac{R}{x} \Rightarrow \phi(\vec{x}) = V\,\frac{R}{x}$$

as one'd expect from Gauss's law.

Consider a conducting sphere in a uniform electric field $\vec{E}_0$: let's guess the answer for ϕ!



$$\phi(\vec{r}) = \underbrace{-\vec{E}_0\cdot\vec{r}}_{\text{potential due to field } \vec{E}_0} + \underbrace{\phi_{sphere}(\vec{r})}_{\text{potential due to the sphere.}}$$

Laplace equation $\nabla^2 \phi = 0$ is valid everywhere outside of the sphere =>

=> $\nabla^2 \phi_{sphere}(\vec{r}) = 0$.

$\phi_{sphere}(\vec{r})$ depends on r and $\vec{E}_0$ & satisfies $\underbrace{\nabla^2 \phi_{sphere} = 0}_{}$ (& vanishes at ∞) => $\phi_{sphere}(\vec{r}) \propto \vec{E}_0 \cdot \vec{\nabla} \frac{1}{r}$

is a natural guess => as $\vec{\nabla} \frac{1}{r} = - \frac{\vec{r}}{r^3}$ =>

$$\phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r} + C \cdot \vec{E}_0 \cdot \frac{\vec{r}}{r^3}$$

constant

Require that $\phi(\vec{r})\Big|_{|\vec{r}|=R} = 0$ => $-1 + \frac{C}{R^3} = 0$

=> $C = R^3$ => $\boxed{\phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r}\left(1 - \frac{R^3}{r^3}\right)}$

(cf. Jackson's (2.14)).

Surface charge density

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial r}\Big|_{r=a} = \varepsilon_0 E_0 \cos\theta + 2\varepsilon_0 E_0 \cos\theta = 3\varepsilon_0 E_0 \cos\theta$$

$$\Rightarrow Q = \oint_S da\, \sigma = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; 3\varepsilon_0 E_0 \cos\theta = 0$$

=> sphere could be insulated or grounded...

We have: ⇒ derived Poisson/Laplace equations

⇒ formulated b.c. problems: Dirichlet & Neumann

⇒ showed that solutions can be expressed in terms of Green functions

⇒ studied one way to find Green functions ~ ~ method of images.

⇒ is there any other way to solve Poisson/Laplace equation? to find Green functions?

Orthogonal Functions

Def.

Orthonormal set of functions is defined by

$$\boxed{\int_a^b dx\, u_n^*(x)\, u_m(x) = \delta_{mn}}$$

where $\delta_{mn} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$ Kronecker delta.

Our goal is to approximate any function

$$f(x) \iff \sum_{n=1}^{N} a_n u_n(x)$$

such that for $N \to \infty$

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x).$$

the coefficients can be determined by multiplying the equality by $u_m^*(x)$ & integrating:

$$\int_a^b dx\, u_m^*(x) f(x) = \sum_{n=1}^{\infty} a_n \underbrace{\int_a^b dx\, u_m^*(x)\, u_n(x)}_{\delta_{mn}} = a_m$$

$$\Rightarrow \quad a_n = \int_a^b dx\, f(x)\, u_n^*(x).$$

Def. The set $\{u_n(x)\}$ is <u>complete</u> if any ("good") function $f(x)$ can be expanded in $\sum_{n=1}^{\infty} a_n u_n(x)$ (or, more precisely, being successfully approximated by partial sums $\sum_{n=1}^{N} a_n u_n(x)$).

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) = \sum_{n=1}^{\infty} \int_a^b dx'\, f(x')\, u_n^*(x')\, u_n(x)$$

$$= \int_a^b dx'\, f(x') \underbrace{\sum_{n=1}^{\infty} u_n^*(x')\, u_n(x)}_{\text{acts like } \delta\text{-ftn},\ \propto \delta(x-x')}$$

$$\Rightarrow \quad \boxed{\sum_{n=1}^{\infty} u_n^*(x')\, u_n(x) = \delta(x-x').}$$

completeness relation.

Famous example: for $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$

$$\{u_n(x)\} \longleftrightarrow \left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right) \right\}$$

Fourier expansion:

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right) \right]$$

where
$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \frac{2}{a} \int_{-a/2}^{a/2} dx\, f(x) \begin{pmatrix} \cos\left(\frac{2\pi m x}{a}\right) \\ \sin\left(\frac{2\pi m x}{a}\right) \end{pmatrix}$$

Check that sines and cosines form complete orthonormal set:

$$\int_{-a/2}^{a/2} dx\, \frac{2}{a} \sin\left(\frac{2\pi m x}{a}\right) \sin\left(\frac{2\pi n x}{a}\right) =$$

$$= \frac{2}{a} \int_{-a/2}^{a/2} dx\, \frac{1}{2} \left[-\cos\left(\frac{2\pi x}{a}(m+n)\right) + \cos\left(\frac{2\pi x}{a}(m-n)\right) \right] =$$

$$= \begin{cases} 1, & m = n \neq 0 \\ 0, & m \neq n,\ m, n > 0 \end{cases}$$

ibid for cosines, etc.

To check completeness let's use complex exponents instead: $u_m(x) = \frac{1}{\sqrt{a}} e^{i\frac{2\pi m x}{a}}$.

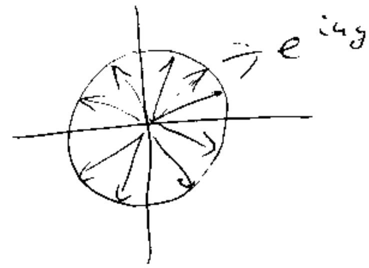
now $m = (-\infty \ldots +\infty)$, $m = 0, \pm 1, \pm 2, \ldots$

Need to see that $\sum_{n=-\infty}^{\infty} u_n^*(x')u_n(x) = \delta(x-x')$

$$\sum_{n=-\infty}^{\infty} u_n^*(x')u_n(x) = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')}$$

Define $y = \frac{2\pi}{a}(x-x') \Rightarrow \frac{1}{a}\sum_{n=-\infty}^{+\infty} e^{in\cdot y} = \begin{cases} 0, & y\neq 0 \\ \infty, & y=0 \end{cases}$

for $y\neq 0$ go to complex plane:



sum of vectors $\simeq 0$.

$\Rightarrow$ condition (i) is satisfied

to check (ii) we integrate

$$\int_{-a/2}^{a/2} dx \cdot \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}\cdot x} = \frac{1}{a}\sum_{n=-\infty}^{\infty}\frac{a}{i2\pi n}\left(e^{i\pi n}-e^{-i\pi n}\right) =$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi n i}\,2i\sin(\pi n) = \sum_{n=-\infty}^{\infty}\frac{\sin(\pi n)}{\pi n} = 1$$

(only $n=0$ contributes)

$\Rightarrow$ the set is complete!

$$f(x) = \sum_m A_m u_m(x) = \frac{1}{\sqrt{a}}\sum_m A_m e^{i\frac{2\pi m x}{a}}$$

<u>Fourier integral:</u> replace $\sum_m \rightarrow \int_{-\infty}^{\infty} dm$

$\frac{2\pi m}{a} \rightarrow k$, $A_m \rightarrow \sqrt{\frac{2\pi}{a}}\,A(k)$ (Jackson's convention)

$$\Rightarrow \boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\cdot x}}$$ Fourier integral / transform

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ik\cdot x}}$$ inverse Fourier transform.

orthogonality condition becomes:

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \delta(k-k')}$$

while the completeness relation is

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')}$$

Let's prove it:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x} = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x - \varepsilon^2 x^2} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\varepsilon^2\left(x - \frac{iq}{2\varepsilon^2}\right)^2}}_{\sqrt{\pi/\varepsilon^2}} e^{-\frac{q^2}{4\varepsilon^2}} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\sqrt{\pi}\,\varepsilon}\, e^{-q^2/4\varepsilon^2} = \frac{1}{2}\,\delta\!\left(\frac{q}{2}\right) = \delta(q)$$

as desired!

representation of δ-ftn studied before