For many dimensions:

$$\delta(\vec{x}-\vec{x}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,e^{ik_x\cdot(x-x')}$$

$$\cdot\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\,e^{ik_y\cdot(y-y')}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\,e^{ik_z\cdot(z-z')} = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

$\Rightarrow$ to solve $\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\,\delta(\vec{x}-\vec{x}\,')$ write

$$G(\vec{x},\vec{x}\,') = \int\frac{d^3k}{(2\pi)^{3/2}}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\cdot\tilde{G}(\vec{k})$$

$$\Rightarrow \nabla^2 G(\vec{x},\vec{x}\,') = \int\frac{d^3k}{(2\pi)^{3/2}}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,(-\vec{k}^2)\,\tilde{G}(\vec{k})$$

$$= -4\pi\,\delta(\vec{x}-\vec{x}\,') = -4\pi\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

$$\Rightarrow -\vec{k}^2\,G(\vec{k}) = -\frac{4\pi}{(2\pi)^{3/2}} \Rightarrow \boxed{G(\vec{k}) = \frac{4\pi}{(2\pi)^{3/2}}\,\frac{1}{\vec{k}^2}}$$

Such that

$$G(\vec{x},\vec{x}\,') = \int\frac{d^3k}{2\pi^2}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,\frac{1}{\vec{k}^2}$$

$\Rightarrow$ going to Fourier space is a powerfull method for solving differential equations (an example of eigenfunction expansion)

Check: $\frac{1}{2\pi^2}\int \frac{d^3k}{\vec{k}^2}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} =$ | go to spherical coordinates, with $\vec{x}-\vec{x}'$ pointing in z-direction

$$= \frac{1}{2\pi^2}\int_0^\infty dk\, \cancel{k^2}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_{-1}^{1} d\cos\theta\, \frac{1}{\cancel{k^2}}\, e^{ik|\vec{x}-\vec{x}'|\cos\theta} =$$

$$= \frac{1}{\pi}\int_0^\infty dk\cdot \frac{1}{ik|\vec{x}-\vec{x}'|}\underbrace{\left[e^{ik\cdot|\vec{x}-\vec{x}'|} - e^{-ik\cdot|\vec{x}-\vec{x}'|}\right]}_{2i\sin(k|\vec{x}-\vec{x}'|)} =$$

$$= \frac{2}{\pi}\,\frac{1}{|\vec{x}-\vec{x}'|}\underbrace{\int_0^\infty \frac{dk}{k}\cdot \sin(k|\vec{x}-\vec{x}'|)}_{=\frac{\pi}{2}\ (\text{see Solution of h/w 1})} = \frac{1}{|\vec{x}-\vec{x}'|} \text{ as desired.}$$

Separation of Variables

A powerfull new tool for solving Poisson/Laplace equations. Depending on geometry we'll consider three main cases: separation of variables in rectangular, spherical & cylindrical coordinates.

Laplace / Poisson Equation in rectangular coordinates

good for problems involving fields / potential in a box. (to get "outside the box" ~ see spherical & cylindrical cases).

Start with Laplace equation in rectangular coordinates: $\nabla^2 \phi = 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Look for solution in the form $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

Plug it in: $X''YZ + XY''Z + XYZ'' = 0$

$$\frac{X''}{X}(x) + \frac{Y''}{Y}(y) + \frac{Z''}{Z}(z) = 0$$

Should work for any $x, y, z \Rightarrow$

$$\left\{\begin{array}{ll} \dfrac{X''}{X} = -\alpha^2 & \Rightarrow X(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \\ \dfrac{Y''}{Y} = -\beta^2 & \Rightarrow Y(y) = \tilde{C}_1 e^{i\beta y} + \tilde{C}_2 e^{-i\beta y} \\ \dfrac{Z''}{Z} = \alpha^2 + \beta^2 \equiv \gamma^2 & \Rightarrow Z(z) = \tilde{\tilde{C}}_1 e^{\gamma z} + \tilde{\tilde{C}}_2 e^{-\gamma z} \end{array}\right.$$

general solution of Laplace eqn.

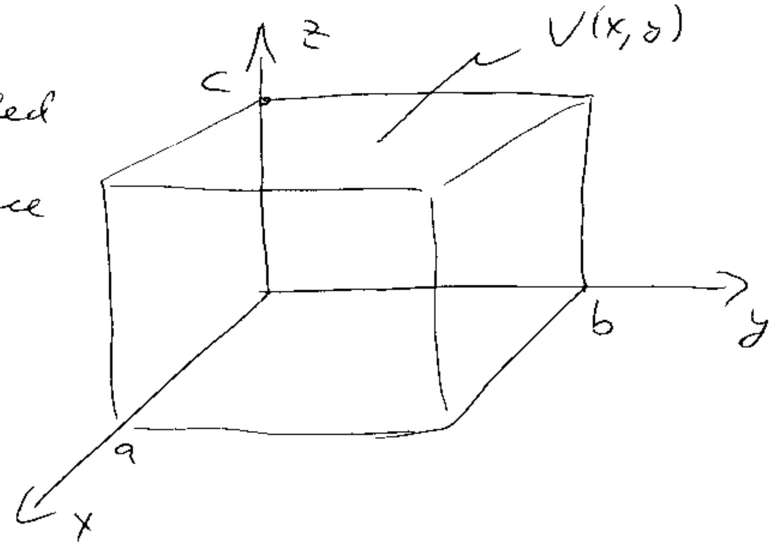
(NB) Eigenfunctions of ∇^2 operator in rectangular coordinates are exponents $e^{\pm a x_i}$, where a can be real or imaginary, and $x_1 = x$, $x_2 = y$, $x_3 = z$.

=> <u>General strategy</u>: use separation of variables to find eigenfunction of ∇^2 operator in various coordinates.

Let's consider an <u>example</u>: a box:

all surfaces grounded exept the top surface sitting at potential $V(x,y)$.

(note: $V(0,y) = V(a,y) = 0$
$V(x,0) = V(x,b) = 0$)



$$X(0) = 0 \Rightarrow X(x) \propto \sin(\alpha x)$$

$$Y(0) = 0 \Rightarrow Y(y) \propto \sin(\beta y)$$

$$Z(0) = 0 \Rightarrow Z(z) \propto \sinh(\gamma z)$$

$$X(a) = 0 \Rightarrow \sin(\alpha a) = 0 \Rightarrow \alpha_n = \frac{\pi n}{a}$$

$$Y(b) = 0 \Rightarrow \beta_n = \frac{\pi m}{b}$$

$$\Rightarrow \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2} = \gamma_{nm}$$

$$\Rightarrow \Phi_{nm}(x,y,z) \propto \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Rightarrow \Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

Finally, $\phi(x,y,z=c) = V(x,y) \Rightarrow$

$$\Rightarrow V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)$$

It's a double Fourier series $\Rightarrow$ can invert obtaining

$$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm} c)} \int_0^{\infty} dx\,dy\, V(x,y)\sin(\alpha_n x)\sin(\beta_m y).$$

problem solved!

To find a general solution for Laplace/Poisson equations with Dirichlet boundary conditions ~~we~~ we need to find Green function $G_D(\vec{x},\vec{x}')$. The construction is similar to using the Fourier transform & we need to solve $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$ and find G_D that vanishes on the boundary.