Method I: expansion in sines:

$$G_D(\vec{x},\vec{x}') \sim \sum_{l,m,n} G_{lmn}(\vec{x}) \sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z'}{c}\right)$$

$G_D(\vec{x},\vec{x}')$ is symmetric under $\vec{x} \leftrightarrow \vec{x}'$ $\Rightarrow$ let's look for it in the form

$$G_D(\vec{x},\vec{x}') = \sum_{l,m,n=-\infty}^{\infty} \frac{8}{abc} G_{lmn} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\cdot$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\cdot \sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right).$$

To solve $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$ we need to find similar representation of δ-ftn.

$$\delta(x-x') = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')} = \frac{1}{a}\sum_{n=-\infty}^{\infty}\cos\left[\frac{2\pi n}{a}(x-x')\right] =$$

$$= \frac{1}{a}\sum_{n=-\infty}^{\infty}\left[\cos\left(\frac{2\pi n}{a}x\right)\cos\left(\frac{2\pi n}{a}x'\right) + \sin\left(\frac{2\pi n}{a}x\right)\sin\left(\frac{2\pi n}{a}x'\right)\right]$$

In the space of functions zero at the boundary, only the sine term contributes: such functions can be expanded into a series of sines:

$$\int_0^a dx\, f(x)\,\delta(x-x') = \int_0^a dx \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n}{a}x\right)\delta(x-x') = \Big| \text{use above formula}$$

$$= \int_0^a dx \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n}{a}x\right)\frac{2}{a}\sum_{m=1}^{\infty}\sin\left(\frac{\pi m}{a}x\right)\sin\left(\frac{\pi m}{a}x'\right) = f(x') \Rightarrow \text{only sines contribute!}$$

$$\nabla^2 G_D(\vec{x},\vec{x}\,') = \sum_{l,m,n} \frac{(-8)}{abc} G_{lmn} \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\pi^2 \cdot$$

$$\cdot \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right) =$$

$$= -4\pi\,\delta^3(\vec{x}-\vec{x}\,') = -4\pi \frac{1}{abc} \sum_{l,m,n} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\ldots$$

(we switched $a \to 2a$ in $\delta^3(\vec{x}-\vec{x}\,')$
$b \to 2b,\ c \to 2c$) See attachment next page

$$\Rightarrow \quad G_{lmn} = \frac{\pi}{2} \frac{1}{\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)} = \frac{1}{2\pi\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)}$$

similar to $\sim \frac{1}{\vec{k}^2}$ in Fourier space

$$\Rightarrow \boxed{G_D(\vec{x},\vec{x}\,') = \frac{4}{\pi abc}\sum_{l,m,n=-\infty}^{\infty} \frac{1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \cdot \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right) \cdot \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}$$

Dirichlet Green Function in a box!

$\Rightarrow$ can use it to find potential $\phi(\vec{x})$ given b.c. on the box

Attachement

We know that on interval $x \in [0, a]$

$$\delta(x) = \frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i \frac{2\pi n}{a} x}$$

$\Rightarrow$ changing $a \to 2a$ we'd like to write

$$" \delta(x) = \frac{1}{2a} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi n}{a} x} ".$$

However, since we're still working in $x \in [0, a]$ interval, we have to fix the norm so that

$$\int_0^a dx \; \delta(x) = 1 \quad \Rightarrow \quad \text{in fact}$$

$$\delta(x) = \frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi n}{a} x}.$$

and, for functions $f(x)$ vanishing at $x = 0$ and $x = a$, $f(0) = f(a) = 0$, we write

$$\delta(x-x') = \frac{1}{a} \sum_{n=-\infty}^{\infty} \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi n}{a} x'\right)$$

Method II: Separation of variables & expansion in hyperbolic sines

By analogy with the solution of the problem of a particle in a box, look for the Green function in the form:

$$G_D(\vec{x},\vec{x}') = \left(\frac{2}{\sqrt{ab}}\right)^2 \sum_{\ell,m=1}^{\infty} g_{\ell m}(z,z') \sin\left(\frac{\ell\pi x}{a}\right)\sin\left(\frac{\ell\pi x'}{a}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)$$

$$\nabla^2 G_D(\vec{x},\vec{x}') = \frac{4}{ab}\sum_{\ell,m=1}^{\infty} \sin\left(\frac{\ell\pi x}{a}\right)\sin\left(\frac{\ell\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y'}{b}\right)\cdot\left\{\left(-\frac{\ell^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}\right) g_{\ell m}(z,z') + \frac{\partial^2}{\partial z^2} g_{\ell m}(z,z')\right\}$$

$$= -4\pi\,\delta^3(\vec{x}-\vec{x}') = -4\pi\,\frac{4}{ab}\sum_{\ell,m=1}^{\infty}\sin\left(\frac{\ell\pi x}{a}\right)\sin\left(\frac{\ell\pi x'}{a}\right)$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right)\cdot\sin\left(\frac{m\pi y'}{b}\right)\,\delta(z-z')$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} g_{\ell m}(z,z') - \pi^2\left(\frac{\ell^2}{a^2}+\frac{m^2}{b^2}\right) g_{\ell m}(z,z') = -4\pi\,\delta(z-z')$$

$$\Rightarrow \text{define } \varkappa_{\ell m} = \sqrt{\pi^2\left(\frac{\ell^2}{a^2}+\frac{m^2}{b^2}\right)} \Rightarrow$$

$$\Rightarrow g_{\ell m}(z,z') = c_1 e^{\varkappa_{\ell m} z} + c_2 e^{-\varkappa_{\ell m} z} \text{ for, say, } z<z'$$

$\Rightarrow$ as $z < z'$ & $g_{\ell m}(0, z') = 0$ (boundary cond'n)

$\Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m} z)$ for $z < z'$

for $z > z'$: $g_{\ell m}(c, z') = 0 \Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m}(z-c))$

$\Rightarrow$ as $g_{\ell m}(z, z') = g_{\ell m}(z', z) \Rightarrow$

$$g_{\ell m}(z, z') \propto \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

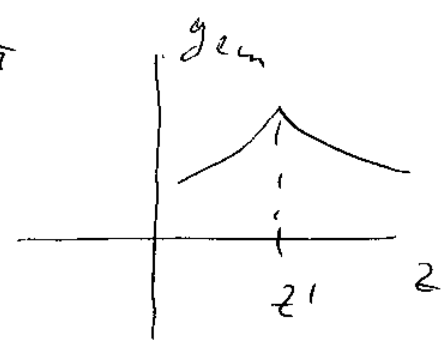
where $z_{\gtrless} = \frac{\max}{\min}\{z, z'\}$.

to take into account the δ-ftn need to integrate over z in the interval $(z'-\varepsilon, z'+\varepsilon)$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = -4\pi$

discontinuity in derivative

(a la Schroedinger equ.)



$$g_{\ell m} = C \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = C(-\varkappa_{\ell m}) \sinh(\varkappa_{\ell m} z') \cdot$

$\cdot \cosh[\varkappa_{\ell m}(c - z')] - C \varkappa_{\ell m} \cosh(\varkappa_{\ell m} z') \sinh[\varkappa_{\ell m}(c - z_>)] =$

$$= -C\,\varkappa_{lm} \sinh\left[\cancel{\varkappa_{lm} z'} + \varkappa_{lm}(c - z')\right] =$$

$$= -C\,\varkappa_{lm} \sinh(\varkappa_{lm} c) = -4\pi$$

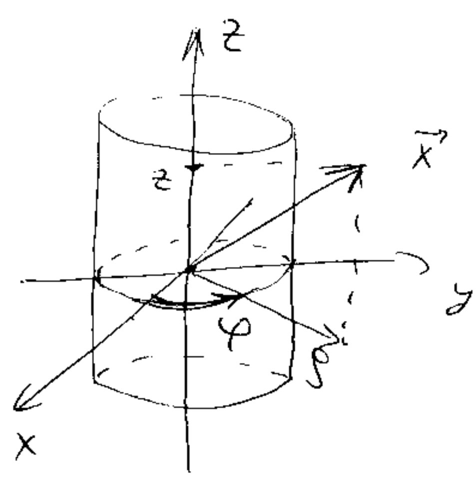
$$\Rightarrow \quad C = \frac{4\pi}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}$$

$$\Rightarrow \boxed{G_D(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot \sin\left(\frac{m\pi y'}{b}\right) \sinh(\varkappa_{lm} z_<) \sinh\left[\varkappa_{lm}(c - z_>)\right] \cdot \frac{1}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}}$$

an alternative decomposition of Green function.

Separation of Variables in Cylindrical Coordinates.

Cylindrical symmetry:



$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$