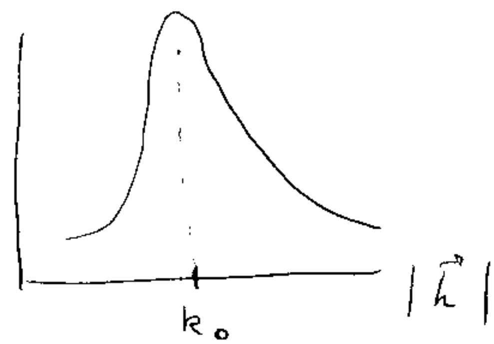
Group and Phase Velocities.

Consider a wave packet:

$$\vec{E}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_0(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

with $E_0(\vec{k})$



$$\Rightarrow \omega(\vec{k}) \approx \omega_0 + (\vec{k}-\vec{k}_0)\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}$$

$$\Rightarrow \vec{k}\cdot\vec{x} - \omega t = \vec{k}\cdot\vec{x} - \omega_0 t - t(\vec{k}-\vec{k}_0)\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0} =$$

$$= \vec{k}\cdot\left(\vec{x} - t(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}\right) \underbrace{- \omega_0 t + t\, \vec{k}_0\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}}_{\text{overall factor (phase)}}$$

$$\Rightarrow \boxed{V_g = \vec{\nabla}_k \omega\Big|_{\vec{k}=\vec{k}_0}} \quad \text{group velocity} \sim$$

$\sim$ speed of the wave packet

cf. $V_{ph} = \frac{\omega}{k}$ ~ phase velocity.

$\Rightarrow$ energy is transferred with V_g, not V_{ph}

Example: $\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow k = \omega\sqrt{\varepsilon\mu_0} = \omega\sqrt{\varepsilon_0\mu_0}\,\cdot$

$$\sqrt{1 - \omega_p^2/\omega^2} = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \Rightarrow V_g = \frac{d\omega}{dk}\Big|_{k=k_0} =$$

$$= \frac{c}{d\sqrt{\omega^2-\omega_p^2}/d\omega} = c\,\frac{\sqrt{\omega^2-\omega_p^2}}{\omega} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}} < c$$

$(\omega > \omega_p)$

$$V_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}} > c$$ violation of relativity?

No, nothing really moves with V_{ph}, all physical quantities move with V_g!

Waveguides and Resonant Cavities.

Waves propagating in confined spaces:

cavity ~ confined in all directions

waveguide ~ confined in all but one direction ~ extended object

Consider a perfect conductor: $\sigma \to \infty$.

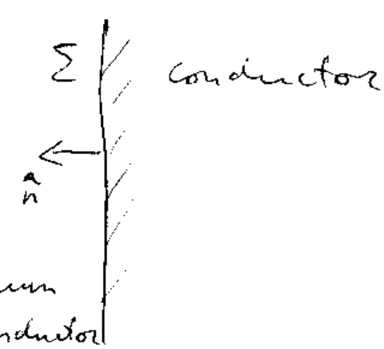
(skin depth $\delta \sim \frac{1}{\sqrt{\sigma}} \to 0$)

$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ and $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = 0$ inside (otherwis $\vec{J}$ infinite)

$\vec{E} = 0 \Rightarrow \rho = 0$ inside $\Rightarrow$ can only have surface density Σ $\Rightarrow$ $\hat{n} \cdot \vec{D} = \Sigma$

as $\vec{E} = 0$ inside $\Rightarrow$ $\hat{n} \times \vec{E} = 0$

(boundary conditions)



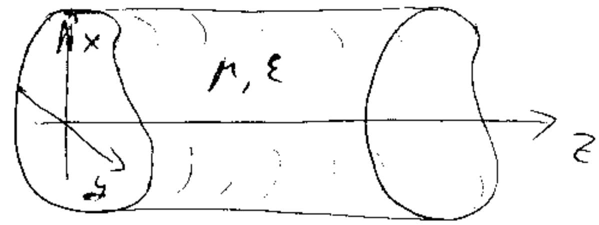
$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \hat{n} \times \vec{H} = \vec{K}$ ~ surface current

($\vec{H} = 0$ inside perfect conductor)

Finally, $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \hat{n} \cdot \vec{B} = 0$.

Wave guides

Maxwell equations:



$$\begin{cases} \vec{\nabla} \times \vec{E} = i\omega \vec{B} & \vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \times \vec{B} = -i\mu\varepsilon\omega \vec{E} & \vec{\nabla} \cdot \vec{B} = 0 \end{cases}$$

perfect conductor outside

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -i\mu\varepsilon\omega \vec{\nabla} \times \vec{E} = \mu\varepsilon\omega^2 \vec{B}$$

$\Rightarrow (\nabla^2 + \mu\varepsilon\omega^2)\vec{B} = 0$. Similarly $(\nabla^2 + \mu\varepsilon\omega^2)\vec{E} = 0$

For waves propagating in the z-direction

$\vec{E}, \vec{B} \sim e^{ikz} \Rightarrow (\nabla^2 + \mu\varepsilon\omega^2)\vec{E} = [\nabla_t^2 + (\mu\varepsilon\omega^2 - k^2)]\vec{E} = 0$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ ~ transverse components

$$\Rightarrow \begin{cases} [\nabla_t^2 + (\mu\varepsilon\omega^2 - k^2)]\vec{E} = 0 \\ [\nabla_t^2 + (\mu\varepsilon\omega^2 - k^2)]\vec{B} = 0 \end{cases}$$

It is convenient to separate transverse and parallel modes: $\vec{E} = \vec{E}_t + \vec{E}_z$, $\vec{E}_z = E_z \cdot \hat{z}$

$$\vec{E}_t = E_x \hat{x} + E_y \hat{y}$$

$\vec{B} = \vec{B}_t + \vec{B}_z$ similarly

Need boundary conditions to solve the above eqn's:

$$\hat{n} \times \vec{E}\Big|_{surface} = 0 \implies E_z\Big|_{surface} = 0 \quad \text{as } \vec{E} = 0 \text{ inside}$$

$$\hat{n} \times (\vec{\nabla} \times \vec{B}) = \cancel{\vec{\nabla}(\hat{n}\cdot\vec{B}) - \vec{\nabla}\vec{B}} = -i\mu\varepsilon\omega\, \vec{\nabla} \times \vec{E}$$

$$\text{''}\ \vec{\nabla}(\hat{n}\cdot\vec{B}) - (\hat{n}\cdot\vec{\nabla})\vec{B}$$

$\hat{n} \cdot \vec{B} = 0$ at the surface, $\vec{\nabla} \times \vec{E} = 0 \implies (\hat{n}\cdot\vec{\nabla})\vec{B} = 0$

$$\implies \frac{\partial B_z}{\partial n}\Big|_{surface} = 0 \quad \text{on the surface}$$

There are different kinds of modes:

Transverse Magnetic modes (TM): $B_z = 0$ everywhere

$$\left(+ \ E_z\Big|_{surface} = 0 \right)$$

Transverse Electric modes (TE): $E_z = 0$ everywhere

$$\left(+ \ \frac{\partial B_z}{\partial n}\Big|_{surface} = 0 \right)$$

Transverse ElectroMagnetic modes (TEM):

$E_z = 0$, $B_z = 0$ everywhere