Magnetic Dipole and Electric Quadrupole

Take the $n=1$ term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int d^3x' \, \vec{J}(\vec{x}')\, \hat{n}\cdot\vec{x}'$$

$$\vec{J}(\hat{n}\cdot\vec{x}') = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$$

as $J_i x_j = \frac{1}{2}(J_i x_j + J_j x_i) + \frac{1}{2}(J_i x_j - J_j x_i)$

(a) take the 2nd term first: remember the magnetization

$\vec{M} = \frac{1}{2}\vec{x}\times\vec{J} \Rightarrow$ the 2nd term gives

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\int d^3x' \, \hat{n}\times\vec{M}(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\hat{n}\times\vec{m}$$ ~ magnetic dipole radiation.

where $\vec{m}$ is the magnetic dipole moment:

$$\vec{m} = \int d^3x' \, \vec{M}(\vec{x}').$$

$\Rightarrow$ can find $\vec{E}, \vec{H}$.

(b) take the 1st term: $\frac{1}{2}\int d^3x' \left[n_i x_i' J_j + n_i J_i x_j'\right] =$

$$= \frac{1}{2}\int d^3x' \, \vec{J}\cdot\vec{\nabla}'\left(x_j'(\hat{n}\cdot\vec{x}')\right) = (\text{parts}) = -\frac{1}{2}\int d^3x' \, x_j'(\hat{n}\cdot\vec{x}')\underbrace{\vec{\nabla}'\cdot\vec{J}}_{i\omega\rho}$$

$$= -\frac{i\omega}{2}\int d^3x' \, x_j'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\frac{\omega k}{2}\frac{e^{ikr}}{r}\cdot\int d^3x'\cdot \vec{x}'\,(\hat{n}\cdot\vec{x}')\,\rho(\vec{x}')$$

electric quadrupole radiation

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{ik}{\mu_0}\,\hat{n}\times\vec{A}\,; \quad \vec{E} = \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{H} =$$

$$= \frac{i}{\omega\varepsilon_0}\,ik\,\hat{n}\times\vec{H} = -\frac{1}{c\varepsilon_0}\frac{ik}{\mu_0}\,\hat{n}\times(\hat{n}\times\vec{A}) = -i\omega\,\hat{n}\times(\hat{n}\times\vec{A})$$

To find $\vec{H}$ need $\hat{n}\times\int d^3x'\,\vec{x}'\,(\hat{n}\cdot\vec{x}')\,\rho(\vec{x}')$.

Quadrupole moment tensor $Q_{\alpha\beta} = \int d^3x\,(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\,\rho$

$$\Rightarrow \left[\hat{n}\times\int d^3x'\,\vec{x}'\,(\hat{n}\cdot\vec{x}')\,\rho(\vec{x}')\right]_i = \varepsilon_{ijk}\,n_j\int d^3x'\cdot x'_k\,\cdot$$

$$\cdot\, n_l\cdot x'_l\;\rho(\vec{x}') = \frac{1}{3}\,\varepsilon_{ijk}\,n_j\,n_l\,Q_{kl} = \frac{1}{3}\,\hat{n}\times\vec{Q}$$

with $(\vec{Q})_\alpha = Q_{\alpha\beta}\,n_\beta \Rightarrow$ $\vec{H} = -\frac{i}{8\pi}\,\omega k^2\,\frac{e^{ikr}}{r}\cdot\frac{1}{3}\,\hat{n}\times\vec{Q}$

$$\vec{E} = -\frac{1}{c\varepsilon_0}\,\hat{n}\times\vec{H} = \frac{i}{24\pi}\,\frac{\omega k^2}{c\varepsilon_0}\,\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{n}\times\vec{Q})$$

Radiated power is (time averaged)

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] = -\frac{1}{2}\,\cancel{r^2}\,\frac{1}{3^2(8\pi)^2}\,\frac{\omega^2k^4}{c\varepsilon_0}\,\frac{1}{\cancel{r^2}}\,\cdot$$

$$\cdot\,\hat{n}\cdot\Big(\big(\hat{n}\times(\hat{n}\times\vec{Q})\big)\times(\hat{n}\times\vec{Q}^*)\Big)$$

$$\hat{n}\cdot\left[\left(\hat{n}\times(\vec{n}\times\vec{Q})\right)\times(\hat{n}\times\vec{Q}^*)\right] = \hat{n}\cdot\left[\left(\hat{n}\,(\hat{n}\cdot\vec{Q})-\vec{Q}\right)\times\right.$$

$$\left.\times\left(\hat{n}\times\vec{Q}^*\right)\right] = \hat{n}\cdot\left[(\hat{n}\cdot\vec{Q})\left(\hat{n}\,(\hat{n}\cdot\vec{Q}^*)-\vec{Q}^*\right)-\hat{n}\,|\vec{Q}|^2+\vec{Q}^*(\hat{n}\cdot\vec{Q})\right]$$

$$= \cancel{(\hat{n}\cdot\vec{Q})^2} - |\vec{Q}|^2 + (\hat{n}\cdot\vec{Q})^2 = -\,Q_{\alpha\beta}\,n_\beta\,Q^*_{\alpha\gamma}\,n_\gamma\,+$$

$$+\;Q_{\alpha\beta}\,n_\alpha n_\beta\;Q^*_{\gamma\delta}\,n_\gamma n_\delta$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c\,k^6}{2(24\pi)^2\,\varepsilon_0}\left(Q_{\alpha\beta}\,n_\beta\,Q^*_{\alpha\gamma}\,n_\gamma - Q_{\alpha\beta}\,n_\alpha n_\beta\,Q^*_{\gamma\delta}\,n_\gamma n_\delta\right)$$

One can integrate this using $Q_{\alpha\alpha}=0$ $\Rightarrow$

$$P = \frac{c\,k^6}{1440\,\pi\,\varepsilon_0}\,|Q_{\alpha\beta}|^2 .$$

<u>Example</u>: ~~spherical~~ ellipsoidal charge distribution

$\Rightarrow Q_{zz} = Q_0, \qquad Q_{xx} = Q_{yy} = -Q_0/2$ as $Q_{\alpha\alpha}=0$

$Q_{ij} = 0$ if $i\neq j$

$$\Rightarrow Q_{\alpha\beta}\,n_\beta \cdot Q_{\alpha\gamma}\,n_\gamma = +\left(\frac{Q_0}{2}\right)^2(n_x^2+n_y^2) + Q_0^2\,n_z^2 =$$

$$= \frac{Q_0^2}{4}\sin^2\theta + Q_0^2\cos^2\theta$$

$$Q_{\alpha\beta}\,n_\alpha n_\beta = -\frac{Q_0}{2}\left(n_x^2+n_y^2\right) + Q_0\,n_z^2 = -\frac{Q_0}{2}\sin^2\theta + Q_0\cos^2\theta$$

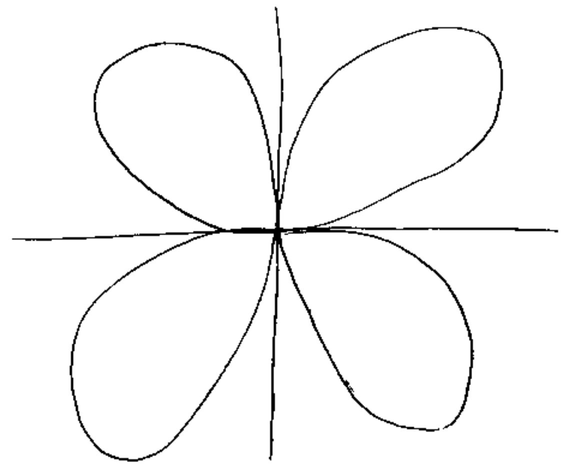
$$\Rightarrow \left|Q_{\alpha\beta}\,n_\alpha n_\beta\right|^2 = \frac{Q_0^2}{4}\sin^4\theta + Q_0^2\cos^4\theta - Q_0^2\sin^2\theta\cos^2\theta$$

$$\Rightarrow Q_{\alpha\beta} n_\beta Q_{\alpha\gamma} n_\gamma - \left(Q_{\alpha\beta} n_\alpha n_\beta \right)^2 = \frac{Q_0^2}{4} \sin^2\theta \cos^2\theta +$$

$$+ Q_0^2 \sin^2\theta \cos^2\theta + Q_0^2 \sin^2\theta \cos^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c k^6 \cdot 9}{2(48\pi)^2 \varepsilon_0} Q_0^2 \sin^2\theta \cos^2\theta}$$

quadrupole radiation pattern:

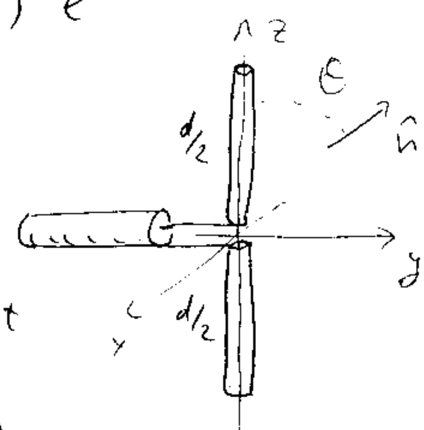


Center - Fed Linear Antenna

In some cases we do not need to expand the vector-potential in the radiation zone:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}\,') \, e^{-ik\hat{n}\cdot\vec{x}\,'}$$

Consider a center-fed linear antena of length d:



$$\vec{J} = I \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\,\delta(y)\,\hat{z} \cdot e^{-i\omega t}$$

vanishes at the ends ($z = \pm d/2$).