# Special Theory of Relativity.

We'll make the following assumptions:

(i) There exist inertial reference frames (Inertial frames move without acceleration, with constant speed, w.r. to each other.)

(ii) Space is uniform/homogeneous

(iii) Space is isotropic. (no preffered direction)

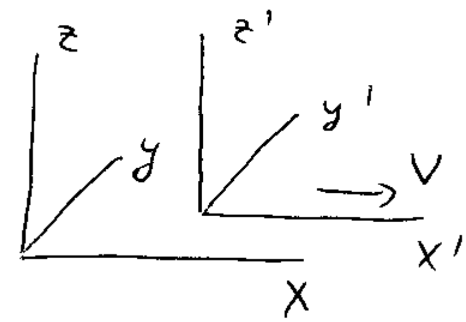
(iv) Time is homogeneous.

Einstein's Postulates:

1. The laws of nature are independent of inertial frame we're in.

2. The speed of light is a finite constant and is independent of inertial frame.

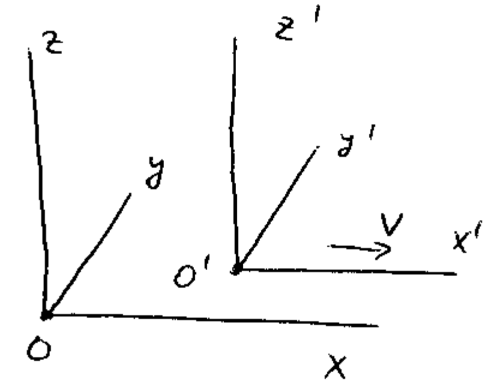
Until 1900's everyone used Galilean transformations:




$$\begin{cases} x' = x - vt \\ t' = t \\ y' = y \quad z' = z \end{cases}$$
(for 2 inertial frames.)

# Lorentz Transformations

Need to find a connection between $(x, y, z, t)$ and $(x', y', z', t')$ for 2 inertial frames moving with speed $V$ w.r.t. each other.




Assume that their origins coinside $(O = O')$ at time $t = t' = 0$.

1) As the space is homogeneous and isotropic

$\Rightarrow \quad y' = ky \quad , \quad z' = kz$

(in general $k = k(\mathbf{v})$)

2) Point $O'$ has $x' = 0$ and $x = Vt$

$\Rightarrow x' = 0$ corresponds to $x - Vt = 0$

$\Rightarrow \quad x' = \alpha(x - Vt) \quad$, where $\alpha = \alpha(V)$

3) Space-time is homogeneous $\Rightarrow$

$\Rightarrow$ write $t' = \delta x + \gamma t \quad$ with $\delta = \delta(v)$, $\gamma = \gamma(v)$

$\Rightarrow$ have 4 unknown parameters:

$\alpha(v)$, $k(v)$, $\delta(v)$, $\gamma(v)$

Choose a coordinate system with $\tilde{x} = -x$, $\tilde{x}' = -x'$
(reverse the $x(x')$ - axis)

$\Rightarrow \quad \tilde{x}' = \alpha(-v)\,(\tilde{x} + vt) \Rightarrow -x' = \alpha(-v)(-x+vt)$

$\Rightarrow x' = \alpha(-v)\,(x - vt) \Rightarrow \alpha(v) = \alpha(-v) \Rightarrow$ write

$\alpha = \alpha(v^2)$

Similarly one can show that $k = k(v^2)$.

$t' = \delta(-v)\,\tilde{x} + \gamma(-v)\,t$

$\Rightarrow t' = -\delta(-v)\,x + \gamma(-v)\,t$

$\Rightarrow$ cf. $\quad t' = \delta(v)\,x + \gamma(v)\,t \Rightarrow \gamma = \gamma(v^2)$

$\delta(-v) = -\delta(v)$.
odd !

Take an inverse transformation &

require that
$$\begin{cases} x = \alpha(-v)(x' + vt') \\ t = \delta(-v)\,x' + \gamma(-v)\,t' \\ y = k(-v)\,y' \\ z = k(-v)\,z' \end{cases}$$

The inverse of
$$\begin{cases} x' = \alpha(x - vt) \\ t' = \delta x + \gamma t \\ y' = ky \\ z' = kz \end{cases} = \begin{pmatrix} \alpha & -\alpha v \\ \delta & \gamma \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

is $x = \dfrac{\gamma x' + \alpha v t'}{\alpha(\gamma + v\delta)}$ , $t = \dfrac{\alpha t' - \delta x'}{\alpha(\gamma + v\delta)}$

$\Rightarrow$ $\alpha(-v) = \alpha(v) = \dfrac{\gamma}{\alpha(\gamma + v\delta)}$ ; $\alpha v = \dfrac{\alpha v}{\alpha(\gamma + v\delta)}$ ;

$\delta(-v) = -\delta(v) = \dfrac{-\delta}{\alpha(\gamma + v\delta)}$ ; $\gamma = \dfrac{1}{\gamma + v\delta}$

$\Rightarrow$ 2nd & 3rd give $\alpha(\gamma + v\delta) = 1$ $\Rightarrow$

$\Rightarrow$ 1st & 4th give $\alpha(v) = \gamma(v)$ . $\Rightarrow$ $\alpha(v) = \dfrac{1}{\alpha(v) + v\delta}$ .

Define a new constant : $\dfrac{\delta(v)}{\alpha(v)} = -v\, f(v^2)$

$\delta$ is an odd function

$\Rightarrow$ $\alpha^2 = \dfrac{1}{1 - v^2 f(v)}$ $\Rightarrow$ $\alpha = \dfrac{1}{\sqrt{1 - v^2 f(v)}} = \gamma$

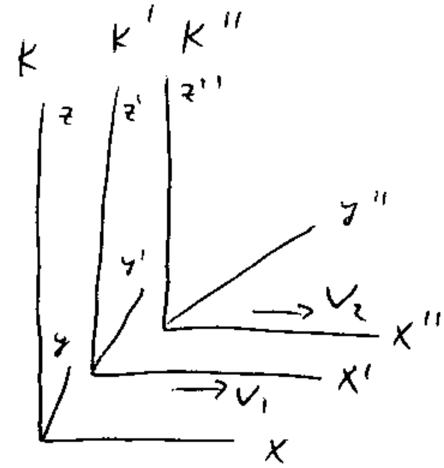
$\delta = -v \alpha f = \dfrac{-vf}{\sqrt{1 - v^2 f}}$

And, finally,

$y = \dfrac{y'}{k} = k(-v) y'$ $\Rightarrow$ as $k(-v) = k(v) \Rightarrow k^2 = 1$

$\Rightarrow$ pick $k = 1$ (need to map onto Galilean transformations at small $v$)

To determine $f$ consider 3 inertial frames $K$, $K'$ & $K''$:




$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & -v_1 \\ -v_1 f_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \alpha_2 \begin{pmatrix} 1 & -v_2 \\ -v_2 f_2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} =$$

$$= \alpha_1 \alpha_2 \begin{pmatrix} 1 & -v_2 \\ -v_2 f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_1 \\ -v_1 f_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \alpha_1 \alpha_2 \cdot$$

$$\cdot \begin{pmatrix} 1 + v_1 v_2 f_1 & -v_1 - v_2 \\ -v_2 f_2 - v_1 f_1 & 1 + v_1 v_2 f_2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$\Rightarrow$ should be also equal to $\alpha(u) \begin{pmatrix} 1 & -u \\ -u f(u) & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

$\Rightarrow f_1 = f_2 \Rightarrow f(v_1) = f(v_2)$ ~ a constant!

Denote $f = \dfrac{1}{c^2}$.

1) $c = \infty \Rightarrow f = 0 \Rightarrow$ get Galilean transformations

2) $c$ ~ finite $\Rightarrow \dfrac{1}{\sqrt{1 - v^2/c^2}}$ ~ real $\Rightarrow$ any $v < c$

$\Rightarrow$ maximum limiting speed!