We know L ⇒ can find canonical momentum

$\vec{P}$ by $$P_i = \frac{\partial L}{\partial V_i} = \underbrace{\gamma m V_i}_{\text{momentum of a free particle}} + \frac{e}{c} A_i = \left(\vec{p} + \frac{e}{c}\vec{A}\right)_i$$

$$\Rightarrow \vec{P} = \vec{p} + \frac{e}{c}\vec{A} \quad \left(\text{in QM } \vec{P} \to -i\hbar\vec{\nabla}\right)$$

quantization

Now we can find the Hamiltonian of the system:

$$H = \vec{P}\cdot\vec{v} - L \Big|_{\text{with } \vec{v} \text{ expressed in terms of } \vec{P}} \Rightarrow$$

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}$$

$$\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 = \frac{m^2 v^2}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow v^2\left(m^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 \frac{1}{c^2}\right) = \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2$$

$$\Rightarrow \vec{v} = \frac{c\left(\vec{P} - \frac{e}{c}\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}.$$

$$\Rightarrow H = \frac{\vec{P}\cdot\left(c\vec{P} - e\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} + e\Phi - \frac{e}{c}\frac{\vec{A}\cdot\left(c\vec{P} - e\vec{A}\right)}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}$$

$$+ mc^2\sqrt{1 - \frac{\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = \frac{c\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} + e\Phi +$$

$$+ \frac{m^2c^3}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\Phi$$

$$\Rightarrow \boxed{H = \sqrt{m^2c^4 + (c\vec{P} - e\vec{A})^2} + e\,\Phi}$$

Motion of a point charge in external $\vec{E}, \vec{B}$ fields:

We have Lorentz force $\dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$

and energy change $\dfrac{dE}{dt} = q\,\vec{v}\cdot\vec{E}$.

A. Constant Electric Field.

$\dfrac{d\vec{p}}{dt} = q\vec{E} \Rightarrow \vec{p} = q\vec{E}t + \text{const} \Rightarrow$ if

the particle starts from rest $\Rightarrow \vec{p}\big|_{t=0} = 0$

$\Rightarrow \vec{p} = q\vec{E}t \Rightarrow$ is $\vec{E} = E\hat{x} \Rightarrow$

$\Rightarrow p_x = qEt \,,\quad p_y = p_z = 0$

$$\Rightarrow \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = qEt \Rightarrow m^2\left(\frac{dx}{dt}\right)^2 = q^2E^2t^2\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t^2}}$$

$$\Rightarrow \quad x(t)\Big| = \int_0^t dt' \, \frac{qEt'}{\sqrt{m^2 + \frac{q^2}{c^2}E^2 t'^2}} = \frac{qE}{m}\,\frac{c^2 m^2}{q^2 E^2}\left(\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2}-1\right)$$

assume $X(0)=0$

$$\Rightarrow \quad x(t) = \frac{mc^2}{qE}\left(\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2}\;-1\right)$$

moves with speed of light!

$\Rightarrow$ as $t\to\infty \Rightarrow x(t) \approx c\;\; t$ ~ linear in t!

$\Rightarrow$ if c is large $\Rightarrow$ expand in powers of $\frac{1}{c}$ $\Rightarrow$

$\Rightarrow x(t) \approx \frac{1}{2}\frac{qE}{m}t^2 = \frac{1}{2}at^2$ ~ well-known classical NR result!

B. Constant Uniform Magnetic Field.

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\,\vec{v}\times\vec{B}\,, \quad \frac{dE}{dt} = 0 \Rightarrow E = \text{const.}$$

$\Rightarrow$ write $\vec{p} = m\gamma\vec{v} = m\gamma c^2\cdot\frac{\vec{v}}{c^2} = E\cdot\frac{\vec{v}}{c^2}$

$$\Rightarrow \frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{q}{c}\,\vec{v}\times\vec{B} \Rightarrow \text{define } \vec{\omega}_B = \frac{q\vec{B}c}{E} = \frac{q\vec{B}}{\gamma mc}$$

(precession frequency)

$$\Rightarrow \frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B \quad \Rightarrow \text{if } \vec{B} = B\hat{z} \Rightarrow \vec{\omega}_B = \omega_B\,\hat{z}$$

$\Rightarrow$ get $\dot{V}_x = \omega_B V_y$, $\dot{V}_y = -\omega_B V_x$, $\dot{V}_z = 0$

$\Rightarrow \ddot{V}_x = \omega_B \dot{V}_y = -\omega_B^2 V_x \Rightarrow V_x = V_{0\perp} \cdot e^{\pm i \omega_B t}$

$\Rightarrow V_y = \frac{1}{\omega_B} \dot{V}_x = \pm i V_{0\perp} e^{\pm i \omega_B t}$

$\Rightarrow$ taking real parts write $V_x = V_{0\perp} \cos(\omega_B t + \alpha)$

$\Rightarrow V_y = -V_{0\perp} \sin(\omega_B t + \alpha) \Rightarrow \sqrt{V_x^2 + V_y^2} = V_{0\perp}$

$\sim$ transverse (w.r.to $\vec{B}$) velocity

$\Rightarrow$ as $V_x = \dot{x} = V_{0\perp} \cos(\omega_B t + \alpha) \Rightarrow$

$$\Rightarrow \begin{cases} x(t) = x_0 + r \sin(\omega_B t + \alpha) \\ y(t) = y_0 + r \cos(\omega_B t + \alpha) \\ z(t) = z_0 + V_{0z} t \end{cases}$$

$$r = \frac{V_{0\perp}}{\omega_B} = \frac{V_{0\perp} E}{qBc} = \frac{c\, p_{0\perp}}{qB}.$$

Motion of a positive charge is shown here $\rightarrow$